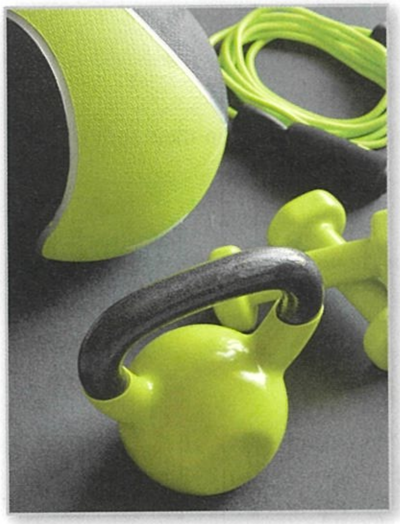
8.2 Operations with Matrices



Matrix operations have many practical applications. For example, in Exercise 80 on page 567, you will use matrix multiplication to analyze the calories burned by individuals of different body weights while performing different types of exercises.

- **Determine whether two matrices are equal.**
- **Add and subtract matrices, and multiply matrices by scalars.**
- **Multiply two matrices.**
- **Use matrices to transform vectors.**
- **Use matrix operations to model and solve real-life problems.**

Equality of Matrices

In Section 8.1, you used matrices to solve systems of linear equations. There is a rich mathematical theory of matrices, and its applications are numerous. This section and the next two sections introduce some fundamental concepts of matrix theory. It is standard mathematical convention to represent matrices in any of the three ways listed below.

Representation of Matrices

1. A matrix can be denoted by an uppercase letter such as A, B, or C.
2. A matrix can be denoted by a representative element enclosed in brackets, such as $[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$.
3. A matrix can be denoted by a rectangular array of numbers such as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** when they have the same dimension $(m \times n)$ and $a_{ij} = b_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$. In other words, two matrices are equal when their corresponding entries are equal.

EXAMPLE 1 Equality of Matrices

Solve for a_{11}, a_{12}, a_{21}, and a_{22} in the matrix equation $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$.

Solution Two matrices are equal when their corresponding entries are equal, so $a_{11} = 2$, $a_{12} = -1$, $a_{21} = -3$, and $a_{22} = 0$.

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Solve for a_{11}, a_{12}, a_{21}, and a_{22} in the matrix equation $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ -2 & 4 \end{bmatrix}$. ■

Be sure you see that for two matrices to be equal, they must have the same dimension *and* their corresponding entries must be equal. For example,

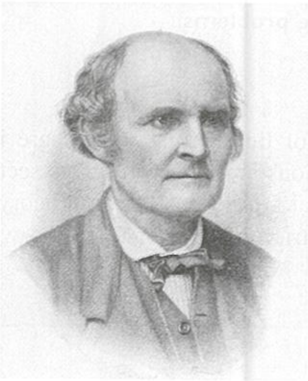
$$\begin{bmatrix} 2 & -1 \\ \sqrt{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 0.5 \end{bmatrix} \quad \text{but} \quad \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & 0 \end{bmatrix} \ne \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}.$$

Angela Aladro mella/Shutterstock.com

Matrix Addition and Scalar Multiplication

Two basic matrix operations are matrix addition and scalar multiplication. With matrix addition, you add two matrices (of the same dimension) by adding their corresponding entries.

HISTORICAL NOTE



Arthur Cayley (1821–1895), a British mathematician, is credited with introducing matrix theory in 1858. Cayley was a Cambridge University graduate and a lawyer by profession. He began his groundbreaking work on matrices as he studied the theory of transformations. Cayley also was instrumental in the development of determinants, which are discussed later in this chapter. Cayley and two American mathematicians, Benjamin Peirce (1809–1880) and his son Charles S. Peirce (1839–1914), are credited with developing "matrix algebra."

Definition of Matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of dimension $m \times n$, then their sum is the $m \times n$ matrix

$$A + B = [a_{ij} + b_{ij}].$$

The sum of two matrices of different dimensions is undefined.

EXAMPLE 2 Addition of Matrices

a. $\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0+(-1) & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 2 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

d. The sum of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & -1 \\ 3 & -2 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

is undefined because A is of dimension 3×3 and B is of dimension 3×2.

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Find each sum, if possible.

a. $\begin{bmatrix} 4 & -1 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 0 & 6 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ -3 & -4 \\ 0 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 9 & 6 \\ 0 & 4 & -2 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 9 & 6 \\ 0 & 2 & -4 \end{bmatrix}$

d. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

In operations with matrices, numbers are usually referred to as **scalars.** In this text, scalars will always be real numbers. To multiply a matrix A by a scalar c, multiply each entry in A by c.

Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar, then the **scalar multiple** of A by c is the $m \times n$ matrix

$$cA = [ca_{ij}].$$

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The symbol $-A$ represents the **negation** of A, which is the scalar product $(-1)A$. Moreover, if A and B are of the same dimension, then $A - B$ represents the sum of A and $(-1)B$. That is,

$$A - B = A + (-1)B. \qquad \text{Subtraction of matrices}$$

EXAMPLE 3 Operations with Matrices

For the matrices below, find (a) $3A$, (b) $-B$, and (c) $3A - B$.

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

Solution

a. $$3A = 3\begin{bmatrix} 2 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \qquad \text{Scalar multiplication}$$

$$= \begin{bmatrix} 3(2) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} \qquad \text{Multiply each entry by 3.}$$

$$= \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} \qquad \text{Simplify.}$$

b. $$-B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} \qquad \text{Definition of negation}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix} \qquad \text{Multiply each entry by } -1.$$

c. $$3A - B = \begin{bmatrix} 6 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix} \qquad 3A - B = 3A + (-1)B$$

$$= \begin{bmatrix} 4 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix} \qquad \text{Add corresponding entries.}$$

REMARK The order of operations for matrix expressions is similar to that for real numbers. As shown in Example 3(c), you perform scalar multiplication before matrix addition and subtraction.

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For the matrices below, find (a) $A - B$, (b) $3A$, and (c) $3A - 2B$.

$$A = \begin{bmatrix} 4 & -1 \\ 0 & 4 \\ -3 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 4 \\ -1 & 3 \\ 1 & 7 \end{bmatrix}$$

It is often convenient to rewrite the scalar multiple cA by factoring c out of every entry in the matrix. The example below shows factoring the scalar $\frac{1}{2}$ out of a matrix.

$$\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{5}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1) & \frac{1}{2}(-3) \\ \frac{1}{2}(5) & \frac{1}{2}(1) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ 5 & 1 \end{bmatrix}$$

ALGEBRA HELP To review the properties of addition and multiplication of real numbers (and other properties of real numbers), see Appendix A.1.

The properties of matrix addition and scalar multiplication are similar to those of addition and multiplication of real numbers.

Properties of Matrix Addition and Scalar Multiplication

Let A, B, and C be $m \times n$ matrices and let c and d be scalars.

1. $A + B = B + A$	Commutative Property of Matrix Addition
2. $A + (B + C) = (A + B) + C$	Associative Property of Matrix Addition
3. $(cd)A = c(dA)$	Associative Property of Scalar Multiplication
4. $1A = A$	Scalar Identity Property
5. $c(A + B) = cA + cB$	Distributive Property
6. $(c + d)A = cA + dA$	Distributive Property

Note that the Associative Property of Matrix Addition allows you to write expressions such as $A + B + C$ without ambiguity because the same sum occurs no matter how the matrices are grouped. This same reasoning applies to sums of four or more matrices.

TECHNOLOGY Most graphing utilities can perform matrix operations. Consult the user's guide for your graphing utility for specific keystrokes. Use a graphing utility to find the sum of the matrices

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1 & 4 \\ 2 & -5 \end{bmatrix}.$$

EXAMPLE 4 Addition of More than Two Matrices

$$\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \qquad \text{Add corresponding entries.}$$

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Evaluate the expression.

$$\begin{bmatrix} 3 & -8 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 6 & -5 \end{bmatrix} + \begin{bmatrix} 0 & 7 \\ 4 & -1 \end{bmatrix}$$

EXAMPLE 5 Evaluating an Expression

$$\begin{aligned} 3\left(\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}\right) &= 3\begin{bmatrix} -2 & 0 \\ 4 & 1 \end{bmatrix} + 3\begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 0 \\ 12 & 3 \end{bmatrix} + \begin{bmatrix} 12 & -6 \\ 9 & 21 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -6 \\ 21 & 24 \end{bmatrix} \end{aligned}$$

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Evaluate the expression.

$$2\left(\begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} -4 & 0 \\ -3 & 1 \end{bmatrix}\right)$$

In Example 5, you could add the two matrices first and then multiply the resulting matrix by 3. The result would be the same.

One important property of addition of real numbers is that the number 0 is the additive identity. That is, $c + 0 = c$ for any real number c. For matrices, a similar property holds. That is, if A is an $m \times n$ matrix and O is the $m \times n$ **zero matrix** consisting entirely of zeros, then

$$A + O = A.$$

In other words, O is the **additive identity** for the set of all $m \times n$ matrices. For example, the matrices below are the additive identities for the sets of all 2×3 and 2×2 matrices.

$$O = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{2 \times 3 \text{ zero matrix}} \quad \text{and} \quad O = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{2 \times 2 \text{ zero matrix}}$$

The algebra of real numbers and the algebra of matrices have many similarities. For example, compare the solutions below.

REMARK When you solve for X in a matrix equation, you are solving for a *matrix* X that makes the equation true.

Real Numbers **(Solve for *x*.)**	**$m \times n$ Matrices** **(Solve for *X*.)**
$x + a = b$	$X + A = B$
$x + a + (-a) = b + (-a)$	$X + A + (-A) = B + (-A)$
$x + 0 = b - a$	$X + O = B - A$
$x = b - a$	$X = B - A$

The algebra of real numbers and the algebra of matrices also have important differences (see Example 9 and Exercises 83–88).

EXAMPLE 6 Solving a Matrix Equation

Solve for X in the equation $3X + A = B$, where

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix}.$$

Solution Begin by solving the matrix equation for X.

$$3X + A = B$$

$$3X = B - A$$

$$X = \tfrac{1}{3}(B - A)$$

Now, substituting the matrices A and B, you have

$$X = \tfrac{1}{3}\left(\begin{bmatrix} -3 & 4 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\right)$$ Substitute the matrices.

$$= \tfrac{1}{3}\begin{bmatrix} -4 & 6 \\ 2 & -2 \end{bmatrix}$$ Subtract matrix A from matrix B.

$$= \begin{bmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$ Multiply the resulting matrix by $\frac{1}{3}$.

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Solve for X in the equation $2X - A = B$, where

$$A = \begin{bmatrix} 6 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}.$$

Matrix Multiplication

Another basic matrix operation is **matrix multiplication.** At first glance, the definition may seem unusual. You will see later, however, that this definition of the product of two matrices has many practical applications.

Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the product AB is an $m \times p$ matrix given by $AB = [c_{ij}]$, where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdot \cdot \cdot + a_{in}b_{nj}$.

$$\underset{m \times n}{A} \times \underset{n \times p}{B} = \underset{m \times p}{AB}$$

Equal

Dimension of AB

The definition of matrix multiplication uses a *row-by-column* multiplication, where the entry in the ith row and jth column of the product AB is obtained by multiplying the entries in the ith row of A by the corresponding entries in the jth column of B and then adding the results. So, for the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of the second matrix. That is, the middle two indices must be the same. The outside two indices give the dimension of the product, as shown at the left. The general pattern for matrix multiplication is shown below.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{ip} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mp} \end{bmatrix}$$

$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdot \cdot \cdot + a_{in}b_{nj} = c_{ij}$$

EXAMPLE 7 Finding the Product of Two Matrices

Find the product AB, where $A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$.

Solution To find the entries of the product, multiply each row of A by each column of B.

$$AB = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)(-3) + 3(-4) & (-1)(2) + 3(1) \\ 4(-3) + (-2)(-4) & 4(2) + (-2)(1) \\ 5(-3) + 0(-4) & 5(2) + 0(1) \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 1 \\ -4 & 6 \\ -15 & 10 \end{bmatrix}$$

REMARK In Example 7, the product AB is defined because the number of columns of A is equal to the number of rows of B. Also, note that the product AB has dimension 3×2.

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Find the product AB, where $A = \begin{bmatrix} -1 & 4 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 \\ 0 & 7 \end{bmatrix}$.

EXAMPLE 8 Finding the Product of Two Matrices

Find the product AB, where $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}$.

Solution Note that the dimension of A is 2×3 and the dimension of B is 3×2. So, the product AB has dimension 2×2.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & 0 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(-2) + 0(1) + 3(-1) & 1(4) + 0(0) + 3(1) \\ 2(-2) + (-1)(1) + (-2)(-1) & 2(4) + (-1)(0) + (-2)(1) \end{bmatrix} \\ &= \begin{bmatrix} -5 & 7 \\ -3 & 6 \end{bmatrix} \end{aligned}$$

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Find the product AB, where $A = \begin{bmatrix} 0 & 4 & -3 \\ 2 & 1 & 7 \\ 3 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 \\ 0 & -4 \\ 1 & 2 \end{bmatrix}$.

EXAMPLE 9 Matrix Multiplication

See LarsonPrecalculus.com for an interactive version of this type of example.

a. $\underset{2 \times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}} \underset{2 \times 2}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \underset{2 \times 2}{\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}}$

b. $\underset{3 \times 3}{\begin{bmatrix} 6 & 2 & 0 \\ 3 & -1 & 2 \\ 1 & 4 & 6 \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}} = \underset{3 \times 1}{\begin{bmatrix} 10 \\ -5 \\ -9 \end{bmatrix}}$

c. $\underset{1 \times 3}{\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}} \underset{3 \times 1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} = \underset{1 \times 1}{\begin{bmatrix} 1 \end{bmatrix}}$

d. $\underset{3 \times 1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} \underset{1 \times 3}{\begin{bmatrix} 1 & -2 & -3 \end{bmatrix}} = \underset{3 \times 3}{\begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}}$

e. The product $\underset{3 \times 2}{\begin{bmatrix} -2 & 1 \\ 1 & -3 \\ 1 & 4 \end{bmatrix}} \underset{3 \times 4}{\begin{bmatrix} -2 & 3 & 1 & 4 \\ 0 & 1 & -1 & 2 \\ 2 & -1 & 0 & 1 \end{bmatrix}}$ is not defined.

REMARK In Examples 9(c) and 9(d), note that the two products are different. Even when both AB and BA are defined, matrix multiplication is not, in general, commutative. That is, for most matrices, $AB \neq BA$. This is one way in which the algebra of real numbers and the algebra of matrices differ.

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Find each product, if possible.

a. $\begin{bmatrix} 1 \\ -3 \end{bmatrix} \begin{bmatrix} 3 & -1 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ **c.** $\begin{bmatrix} 3 & 1 & 2 \\ 7 & 0 & -2 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ 2 & -1 \end{bmatrix}$

EXAMPLE 10 Squaring a Matrix

Find A^2, where $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. (*Note:* $A^2 = AA$.)

Solution

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

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Find A^2, where $A = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$.

Properties of Matrix Multiplication

Let A, B, and C be matrices and let c be a scalar.

1. $A(BC) = (AB)C$ Associative Property of Matrix Multiplication

2. $A(B + C) = AB + AC$ Left Distributive Property

3. $(A + B)C = AC + BC$ Right Distributive Property

4. $c(AB) = (cA)B = A(cB)$ Associative Property of Scalar Multiplication

Definition of the Identity Matrix

The $n \times n$ matrix that consists of 1's on its main diagonal and 0's elsewhere is called the **identity matrix of dimension $n \times n$** and is denoted by

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$ Identity matrix

Note that an identity matrix must be *square*. When the dimension is understood to be $n \times n$, you can denote I_n simply by I.

If A is an $n \times n$ matrix, then the identity matrix has the property that $AI_n = A$ and $I_nA = A$. For example,

$$\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix}$$ $AI = A$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 4 \\ -1 & 2 & -3 \end{bmatrix}.$$ $IA = A$

Using Matrices to Transform Vectors

In Section 6.3, you performed vector operations with vectors written in component form and with vectors written as linear combinations of the standard unit vectors **i** and **j**. Another way to perform vector operations is with the vectors written as column matrices.

EXAMPLE 11 Vector Operations

Let $\mathbf{v} = \langle 2, 4\rangle$ and $\mathbf{w} = \langle 6, 2\rangle$. Use matrices to find each vector.

a. $\mathbf{v} + \mathbf{w}$ **b.** $\mathbf{w} - 2\mathbf{v}$

Solution Begin by writing **v** and **w** as column matrices.

$$\mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

a. $\mathbf{v} + \mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \langle 8, 6\rangle$

Figure 8.1 shows a sketch of $\mathbf{v} + \mathbf{w}$.

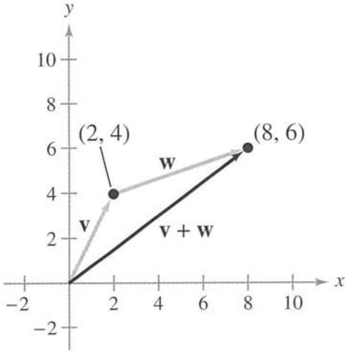

Figure 8.1

b. $\mathbf{w} - 2\mathbf{v} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \langle 2, -6\rangle$

Figure 8.2 shows a sketch of $\mathbf{w} - 2\mathbf{v} = \mathbf{w} + (-2\mathbf{v})$.

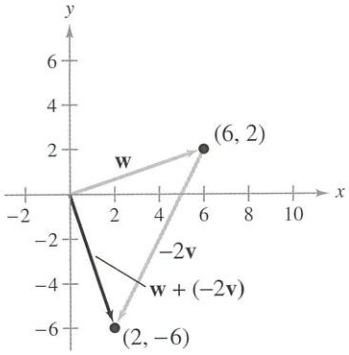

Figure 8.2

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Let $\mathbf{v} = \langle 3, 6\rangle$ and $\mathbf{w} = \langle 8, 5\rangle$. Use matrices to find each vector.

a. $\mathbf{v} - \mathbf{w}$ **b.** $3\mathbf{v} + \mathbf{w}$

One way to transform a vector **v** is to multiply **v** by a square **transformation matrix** A to produce another vector $A\mathbf{v}$. A column matrix with two rows can represent a vector **v**, so the transformation matrix must have two columns (and also two rows) for $A\mathbf{v}$ to be defined.

EXAMPLE 12 Describing a Vector Transformation

Find the product $A\mathbf{v}$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\mathbf{v} = \langle 1, 3\rangle$, and describe the transformation.

Solution First note that A has two columns and **v**, written as the column matrix $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, has two rows, so $A\mathbf{v}$ is defined.

$$A\mathbf{v} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \langle 1, -3\rangle$$

Figure 8.3 shows a sketch of the vectors **v** and $A\mathbf{v}$. The matrix A transforms **v** by reflecting **v** in the x-axis.

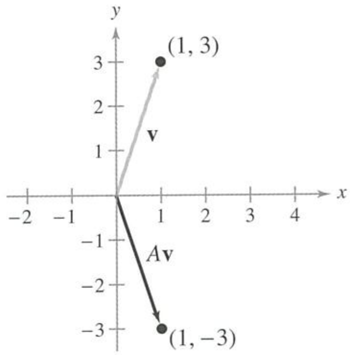

Figure 8.3

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Find the product $A\mathbf{v}$, where $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{v} = \langle 3, 1\rangle$, and describe the transformation.

Applications

Matrix multiplication can be used to represent a system of linear equations. Note how the system below can be written as the matrix equation $AX = B$, where A is the *coefficient matrix* of the system and X and B are column matrices. The column matrix B is also called a *constant matrix*. Its entries are the constant terms in the system of equations.

System

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$$

Matrix Equation $AX = B$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underset{\times}{} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} \underset{=}{=} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{B}$$

In Example 13, $[A \,\vdots\, B]$ represents the augmented matrix formed when you *adjoin* matrix B to matrix A. Also, $[I \,\vdots\, X]$ represents the reduced row-echelon form of the augmented matrix that yields the solution of the system.

Many real-life applications of linear systems involve enormous numbers of equations and variables. For example, a flight crew scheduling problem for American Airlines required the manipulation of matrices with 837 rows and 12,753,313 columns. *(Source: Very Large-Scale Linear Programming. A Case Study in Combining Interior Point and Simplex Methods, Bixby, Robert E., et al., Operations Research, 40, no. 5)*

EXAMPLE 13 Solving a System of Linear Equations

For the system of linear equations, (a) write the system as a matrix equation, $AX = B$, and (b) use Gauss-Jordan elimination on $[A \,\vdots\, B]$ to solve for the matrix X.

$$\begin{cases} x_1 - 2x_2 + x_3 = -4 \\ x_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 - 2x_3 = 2 \end{cases}$$

Solution

a. In matrix form, $AX = B$, the system is

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 2 \end{bmatrix}.$$

b. Form the augmented matrix by adjoining matrix B to matrix A.

$$[A \,\vdots\, B] = \left[\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & 1 & 2 & 4 \\ 2 & 3 & -2 & 2 \end{array}\right]$$

Using Gauss-Jordan elimination, rewrite this matrix as

$$[I \,\vdots\, X] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

So, the solution of the matrix equation is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

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For the system of linear equations, (a) write the system as a matrix equation, $AX = B$, and (b) use Gauss-Jordan elimination on $[A \,\vdots\, B]$ to solve for the matrix X.

$$\begin{cases} -2x_1 - 3x_2 = -4 \\ 6x_1 + x_2 = -36 \end{cases}$$

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EXAMPLE 14 Softball Team Expenses

Two softball teams submit equipment lists to their sponsors.

Equipment	**Women's Team**	**Men's Team**
Bats	12	15
Balls	45	38
Gloves	15	17

Each bat costs \$80, each ball costs \$4, and each glove costs \$90. Use matrices to find the total cost of equipment for each team.

Solution Write the equipment lists E and the costs per item C in matrix form as

$$E = \begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 80 & 4 & 90 \end{bmatrix}.$$

REMARK Notice in Example 14 that it is not possible to find the total cost using the product EC because EC is not defined. That is, the number of columns of E (2 columns) does not equal the number of rows of C (1 row).

The total cost of equipment for each team is the product

$$\begin{aligned} CE &= \begin{bmatrix} 80 & 4 & 90 \end{bmatrix} \begin{bmatrix} 12 & 15 \\ 45 & 38 \\ 15 & 17 \end{bmatrix} \\ &= \begin{bmatrix} 80(12) + 4(45) + 90(15) & 80(15) + 4(38) + 90(17) \end{bmatrix} \\ &= \begin{bmatrix} 2490 & 2882 \end{bmatrix}. \end{aligned}$$

So, the total cost of equipment for the women's team is \$2490, and the total cost of equipment for the men's team is \$2882.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Repeat Example 14 when each bat costs \$100, each ball costs \$3, and each glove costs \$65.

Summarize (Section 8.2)

1. State the conditions under which two matrices are equal *(page 553)*. For an example involving matrix equality, see Example 1.
2. Explain how to add matrices *(page 554)*. For an example of matrix addition, see Example 2.
3. Explain how to multiply a matrix by a scalar *(page 554)*. For an example of scalar multiplication, see Example 3.
4. List the properties of matrix addition and scalar multiplication *(page 556)*. For examples of using these properties, see Examples 4–6.
5. Explain how to multiply two matrices *(page 558)*. For examples of matrix multiplication, see Examples 7–10.
6. Explain how to use matrices to transform vectors *(page 561)*. For examples involving matrices and vectors, see Examples 11 and 12.
7. Describe real-life applications of matrix operations *(pages 562 and 563, Examples 13 and 14)*.

8.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. Two matrices are ________ when their corresponding entries are equal.
2. When performing matrix operations, real numbers are usually referred to as ________.
3. A matrix consisting entirely of zeros is called a ________ matrix and is denoted by ________.
4. The $n \times n$ matrix that consists of 1's on its main diagonal and 0's elsewhere is called the ________ matrix of dimension $n \times n$.

Skills and Applications

Equality of Matrices **In Exercises 5–8, solve for x and y.**

5. $\begin{bmatrix} x & -2 \\ 7 & 23 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ 7 & y \end{bmatrix}$ **6.** $\begin{bmatrix} -5 & x \\ 3y & 8 \end{bmatrix} = \begin{bmatrix} -5 & 13 \\ 12 & 8 \end{bmatrix}$

7. $\begin{bmatrix} 16 & 4 & x & 4 \\ 0 & 2 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 4 & 2x+1 & 4 \\ 0 & 2 & 3y-5 & 0 \end{bmatrix}$

8. $\begin{bmatrix} x+2 & 8 & -3 \\ 1 & 18 & -8 \\ 7 & -2 & y+2 \end{bmatrix} = \begin{bmatrix} 2x+6 & 8 & -3 \\ 1 & 18 & -8 \\ 7 & -2 & x \end{bmatrix}$

Operations with Matrices **In Exercises 9–16, if possible, find (a) $A + B$, (b) $A - B$, (c) $3A$, and (d) $3A - 2B$.**

9. $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

10. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & -2 \\ 4 & 2 \end{bmatrix}$

11. $A = \begin{bmatrix} 6 & 0 & 3 \\ -1 & -4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 8 & -1 \\ 4 & -3 \end{bmatrix}$

12. $A = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $B = \begin{bmatrix} -4 & 6 & 2 \end{bmatrix}$

13. $A = \begin{bmatrix} 8 & -1 \\ 2 & 3 \\ -4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 6 \\ -1 & -5 \\ 1 & 10 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 6 & 9 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 0 & -5 \\ -3 & 4 & -7 \end{bmatrix}$

15. $A = \begin{bmatrix} 4 & 5 & -1 & 3 & 4 \\ 1 & 2 & -2 & -1 & 0 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 \\ -6 & 8 & 2 & -3 & -7 \end{bmatrix}$

16. $A = \begin{bmatrix} -1 & 4 & 0 \\ 3 & -2 & 2 \\ 5 & 4 & -1 \\ 0 & 8 & -6 \\ -4 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 5 & 1 \\ 2 & -4 & -7 \\ 10 & -9 & -1 \\ 3 & 2 & -4 \\ 0 & 1 & -2 \end{bmatrix}$

Evaluating an Expression **In Exercises 17–22, evaluate the expression.**

17. $\begin{bmatrix} -5 & 0 \\ 3 & -6 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} -10 & -8 \\ 14 & 6 \end{bmatrix}$

18. $\begin{bmatrix} 6 & 8 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ -3 & -1 \end{bmatrix} + \begin{bmatrix} -11 & -7 \\ 2 & -1 \end{bmatrix}$

19. $4\left(\begin{bmatrix} -4 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -2 \\ 3 & -6 & 0 \end{bmatrix}\right)$

20. $\frac{1}{2}(\begin{bmatrix} 5 & -2 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 14 & 6 & -18 & 9 \end{bmatrix})$

21. $-3\left(\begin{bmatrix} 0 & -3 \\ 7 & 2 \end{bmatrix} + \begin{bmatrix} -6 & 3 \\ 8 & 1 \end{bmatrix}\right) - 2\begin{bmatrix} 4 & -4 \\ 7 & -9 \end{bmatrix}$

22. $-1\begin{bmatrix} 4 & 11 \\ -2 & -1 \\ 9 & 3 \end{bmatrix} + \frac{1}{6}\left(\begin{bmatrix} -5 & -1 \\ 3 & 4 \\ 0 & 13 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ -9 & -1 \\ 6 & -1 \end{bmatrix}\right)$

Operations with Matrices **In Exercises 23–26, use the matrix capabilities of a graphing utility to evaluate the expression.**

23. $\frac{11}{25}\begin{bmatrix} 2 & 5 \\ -1 & -4 \end{bmatrix} + 6\begin{bmatrix} -3 & 0 \\ 2 & 2 \end{bmatrix}$

24. $55\left(\begin{bmatrix} 14 & -11 \\ -22 & 19 \end{bmatrix} - \begin{bmatrix} -8 & 20 \\ 13 & 6 \end{bmatrix}\right)$

25. $-2\begin{bmatrix} 1.23 & 4.19 & -3.85 \\ 7.21 & -2.60 & 6.54 \end{bmatrix} - \begin{bmatrix} 8.35 & -3.02 & 7.30 \\ -0.38 & -5.49 & 1.68 \end{bmatrix}$

26. $-1\begin{bmatrix} 10 & 15 \\ -20 & 10 \\ 12 & 4 \end{bmatrix} + \frac{1}{8}\left(\begin{bmatrix} -13 & 11 \\ 7 & 0 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} -3 & 13 \\ -3 & 8 \\ -14 & 15 \end{bmatrix}\right)$

Solving a Matrix Equation **In Exercises 27–34, solve for X in the equation, where**

$$A = \begin{bmatrix} -2 & 1 & 3 \\ -1 & 0 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2 & -4 \\ 3 & 0 & 1 \end{bmatrix}.$$

27. $X = 2A + 2B$ **28.** $X = 3A - 2B$

29. $2X = 2A - B$ **30.** $2X = A + B$

31. $2X + 3A = B$ **32.** $3X - 4A = 2B$

33. $4B = -2X - 2A$ **34.** $5A = 6B - 3X$

Finding the Product of Two Matrices **In Exercises 35–40, if possible, find AB and state the dimension of the result.**

35. $A = \begin{bmatrix} -1 & 6 \\ -4 & 5 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 0 & 9 \end{bmatrix}$

36. $A = \begin{bmatrix} 0 & -1 & 2 \\ 6 & 0 & 3 \\ 7 & -1 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 4 & -5 \\ 1 & 6 \end{bmatrix}$

37. $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \\ 1 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & 2 \\ 8 & -1 & 7 \end{bmatrix}$

38. $A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 6 & 13 & 8 & -17 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 6 \\ 4 & 2 \end{bmatrix}$

39. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{8} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

40. $A = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & -3 \\ 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -11 & 4 \\ 8 & 16 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

Finding the Product of Two Matrices **In Exercises 41–44, use the matrix capabilities of a graphing utility to find AB, if possible.**

41. $A = \begin{bmatrix} 7 & 5 & -4 \\ -2 & 5 & 1 \\ 10 & -4 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 & 3 \\ 8 & 1 & 4 \\ -4 & 2 & -8 \end{bmatrix}$

42. $A = \begin{bmatrix} 11 & -12 & 4 \\ 14 & 10 & 12 \\ 6 & -2 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 12 & 10 \\ -5 & 12 \\ 15 & 16 \end{bmatrix}$

43. $A = \begin{bmatrix} -3 & 8 & -6 & 8 \\ -12 & 15 & 9 & 6 \\ 5 & -1 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 6 \\ 24 & 15 & 14 \\ 16 & 10 & 21 \\ 8 & -4 & 10 \end{bmatrix}$

44. $A = \begin{bmatrix} -2 & 4 & 8 \\ 21 & 5 & 6 \\ 13 & 2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ -7 & 15 \\ 32 & 14 \\ 0.5 & 1.6 \end{bmatrix}$

Operations with Matrices **In Exercises 45–52, if possible, find (a) AB, (b) BA, and (c) A^2.**

45. $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -1 & 8 \end{bmatrix}$

46. $A = \begin{bmatrix} 6 & 3 \\ -2 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 0 \\ 2 & 4 \end{bmatrix}$

47. $A = \begin{bmatrix} 5 & -9 & 0 \\ 3 & 0 & -8 \\ -1 & 4 & 11 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

48. $A = \begin{bmatrix} 2 & -2 \\ -3 & 0 \\ 7 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

49. $A = \begin{bmatrix} -4 & -1 \\ 2 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$

50. $A = \begin{bmatrix} 1 & 3 & -2 \\ -5 & 10 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

51. $A = \begin{bmatrix} 7 \\ 8 \\ -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$

52. $A = \begin{bmatrix} 3 & 2 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$

Operations with Matrices **In Exercises 53–56, evaluate the expression. Use the matrix capabilities of a graphing utility to verify your answer.**

53. $\begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$

54. $-3\left(\begin{bmatrix} 6 & 5 & -1 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -1 & -3 \\ 4 & 1 \end{bmatrix}\right)$

55. $\begin{bmatrix} 0 & 2 & -2 \\ 4 & 1 & 2 \end{bmatrix}\left(\begin{bmatrix} 4 & 0 \\ 0 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ -3 & 5 \\ 0 & -3 \end{bmatrix}\right)$

56. $\begin{bmatrix} 3 \\ -1 \\ 5 \\ 7 \end{bmatrix}\left(\begin{bmatrix} 5 & -6 \end{bmatrix} + \begin{bmatrix} 7 & -1 \end{bmatrix} + \begin{bmatrix} -8 & 9 \end{bmatrix}\right)$

Vector Operations **In Exercises 57–60, use matrices to find (a) $\mathbf{u} + \mathbf{v}$, (b) $\mathbf{u} - \mathbf{v}$, and (c) $3\mathbf{v} - \mathbf{u}$.**

57. $\mathbf{u} = \langle 1, 5 \rangle, \quad \mathbf{v} = \langle 3, 2 \rangle$

58. $\mathbf{u} = \langle 4, 2 \rangle, \quad \mathbf{v} = \langle 6, -3 \rangle$

59. $\mathbf{u} = \langle -2, 2 \rangle, \quad \mathbf{v} = \langle 5, 4 \rangle$

60. $\mathbf{u} = \langle 7, -4 \rangle, \quad \mathbf{v} = \langle 2, 1 \rangle$

Describing a Vector Transformation **In Exercises 61–66, find $A\mathbf{v}$, where $\mathbf{v} = \langle 4, 2 \rangle$, and describe the transformation.**

61. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

62. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

63. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

64. $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

65. $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

66. $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Solving a System of Linear Equations

In Exercises 67–72, (a) write the system of linear equations as a matrix equation, $AX = B$, and (b) use Gauss-Jordan elimination on $[A \,\vdots\, B]$ to solve for the matrix X.

67. $\begin{cases} 2x_1 + 3x_2 = 5 \\ x_1 + 4x_2 = 10 \end{cases}$

68. $\begin{cases} -2x_1 - 3x_2 = -4 \\ 6x_1 + x_2 = -36 \end{cases}$

69. $\begin{cases} x_1 - 2x_2 + 3x_3 = 9 \\ -x_1 + 3x_2 - x_3 = -6 \\ 2x_1 - 5x_2 + 5x_3 = 17 \end{cases}$

70. $\begin{cases} x_1 + x_2 - 3x_3 = -1 \\ -x_1 + 2x_2 = 1 \\ x_1 - x_2 + x_3 = 2 \end{cases}$

71. $\begin{cases} x_1 - 5x_2 + 2x_3 = -20 \\ -3x_1 + x_2 - x_3 = 8 \\ -2x_2 + 5x_3 = -16 \end{cases}$

72. $\begin{cases} x_1 - x_2 + 4x_3 = 17 \\ x_1 + 3x_2 = -11 \\ -6x_2 + 5x_3 = 40 \end{cases}$

73. **Manufacturing** A corporation has four factories that manufacture sport utility vehicles and pickup trucks. The production levels are represented by A.

$$A = \begin{array}{c@{}l} \overbrace{\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}}^{\text{Factory}} & \\ \begin{bmatrix} 100 & 90 & 70 & 30 \\ 40 & 20 & 60 & 60 \end{bmatrix} & \begin{array}{l} \text{SUV} \\ \text{Pickup} \end{array} \Big\} \begin{array}{l} \text{Vehicle} \\ \text{Type} \end{array} \end{array}$$

Find the production levels when production increases by 10%.

74. **Vacation Packages** A travel agent identifies four resorts with special all-inclusive packages. The current rates for two types of rooms (double and quadruple occupancy) at the four resorts are represented by A.

$$A = \begin{array}{c@{}l} \begin{array}{cccc} \text{Resort } w & \text{Resort } x & \text{Resort } y & \text{Resort } z \end{array} & \\ \begin{bmatrix} 615 & 670 & 740 & 990 \\ 995 & 1030 & 1180 & 1105 \end{bmatrix} & \begin{array}{l} \text{Double} \\ \text{Quadruple} \end{array} \Big\} \text{Occupancy} \end{array}$$

The rates are expected to increase by no more than 12% by next season. Find the maximum rate per package per resort.

75. **Agriculture** A farmer grows apples and peaches. Each crop is shipped to three different outlets. The shipment levels are represented by A.

$$A = \begin{array}{c@{}l} \overbrace{\begin{array}{ccc} 1 & 2 & 3 \end{array}}^{\text{Outlet}} & \\ \begin{bmatrix} 125 & 100 & 75 \\ 100 & 175 & 125 \end{bmatrix} & \begin{array}{l} \text{Apples} \\ \text{Peaches} \end{array} \Big\} \text{Crop} \end{array}$$

The profits per unit are represented by the matrix $B = [\$3.50 \quad \$6.00]$. Compute BA and interpret the result.

76. **Revenue** An electronics manufacturer produces three models of high-definition televisions, which are shipped to two warehouses. The shipment levels are represented by A.

$$A = \begin{array}{c@{}l} \overbrace{\begin{array}{cc} 1 & 2 \end{array}}^{\text{Warehouse}} & \\ \begin{bmatrix} 5{,}000 & 4{,}000 \\ 6{,}000 & 10{,}000 \\ 8{,}000 & 5{,}000 \end{bmatrix} & \begin{array}{l} \text{A} \\ \text{B} \\ \text{C} \end{array} \Bigg\} \text{Model} \end{array}$$

The prices per unit are represented by the matrix

$$B = [\$699.95 \quad \$899.95 \quad \$1099.95].$$

Compute BA and interpret the result.

77. **Labor and Wages** A company has two factories that manufacture three sizes of boats. The numbers of hours of labor required to manufacture each size are represented by S.

$$S = \begin{array}{c@{}l} \overbrace{\begin{array}{ccc} \text{Cutting} & \text{Assembly} & \text{Packaging} \end{array}}^{\text{Department}} & \\ \begin{bmatrix} 1.0 & 0.5 & 0.2 \\ 1.6 & 1.0 & 0.2 \\ 2.5 & 2.0 & 1.4 \end{bmatrix} & \begin{array}{l} \text{Small} \\ \text{Medium} \\ \text{Large} \end{array} \Bigg\} \text{Boat size} \end{array}$$

The wages of the workers are represented by T.

$$T = \begin{array}{c@{}l} \overbrace{\begin{array}{cc} \text{A} & \text{B} \end{array}}^{\text{Factory}} & \\ \begin{bmatrix} \$15 & \$13 \\ \$12 & \$11 \\ \$11 & \$10 \end{bmatrix} & \begin{array}{l} \text{Cutting} \\ \text{Assembly} \\ \text{Packaging} \end{array} \Bigg\} \text{Department} \end{array}$$

Compute ST and interpret the result.

78. **Profit** At a local store, the numbers of gallons of skim milk, 2% milk, and whole milk sold over the weekend are represented by A.

$$A = \begin{array}{c@{}l} \begin{array}{ccc} \text{Skim milk} & \text{2\% milk} & \text{Whole milk} \end{array} & \\ \begin{bmatrix} 40 & 64 & 52 \\ 60 & 82 & 76 \\ 76 & 96 & 84 \end{bmatrix} & \begin{array}{l} \text{Friday} \\ \text{Saturday} \\ \text{Sunday} \end{array} \end{array}$$

The selling prices per gallon and the profits per gallon for the three types of milk are represented by B.

$$B = \begin{array}{c@{}l} \begin{array}{cc} \text{Selling price} & \text{Profit} \end{array} & \\ \begin{bmatrix} \$3.45 & \$1.20 \\ \$3.65 & \$1.30 \\ \$3.85 & \$1.45 \end{bmatrix} & \begin{array}{l} \text{Skim milk} \\ \text{2\% milk} \\ \text{Whole milk} \end{array} \end{array}$$

(a) Compute AB and interpret the result.

(b) Find the store's total profit from milk sales for the weekend.

79. Voting Preferences The matrix

$$\begin{array}{rc} & \overbrace{\begin{array}{ccc}\text{R} & \text{D} & \text{I}\end{array}}^{\text{From}} \\ P = & \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{bmatrix} \end{array} \left.\begin{array}{l}\text{R}\\ \text{D}\\ \text{I}\end{array}\right\} \text{To}$$

is called a *stochastic matrix.* Each entry p_{ij} $(i \neq j)$ represents the proportion of the voting population that changes from party i to party j, and p_{ii} represents the proportion that remains loyal to the party from one election to the next. Compute and interpret P^2.

80. Exercise The numbers of calories burned by individuals of different body weights while performing different types of exercises for a one-hour time period are represented by A.

$$\begin{array}{rc} & \overbrace{\begin{array}{cc}\text{130-lb person} & \text{155-lb person}\end{array}}^{\text{Calories burned}} \\ A = & \begin{bmatrix} 472 & 563 \\ 590 & 704 \\ 177 & 211 \end{bmatrix} \end{array} \begin{array}{l}\text{Basketball}\\ \text{Jumping rope}\\ \text{Weight lifting}\end{array}$$

(a) A 130-pound person and a 155-pound person play basketball for 2 hours, jump rope for 15 minutes, and lift weights for 30 minutes. Organize the times spent exercising in a matrix B.

(b) Compute BA and interpret the result.

Exploration

True or False? **In Exercises 81 and 82, determine whether the statement is true or false. Justify your answer.**

81. Two matrices can be added only when they have the same dimension.

82. Matrix multiplication is commutative.

Think About It **In Exercises 83–86, use the matrices**

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}.$$

83. Show that $(A + B)^2 \neq A^2 + 2AB + B^2$.

84. Show that $(A - B)^2 \neq A^2 - 2AB + B^2$.

85. Show that $(A + B)(A - B) \neq A^2 - B^2$.

86. Show that $(A + B)^2 = A^2 + AB + BA + B^2$.

87. Think About It If a, b, and c are real numbers such that $c \neq 0$ and $ac = bc$, then $a = b$. However, if A, B, and C are nonzero matrices such that $AC = BC$, then A is *not necessarily* equal to B. Illustrate this using the following matrices.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

88. Think About It If a and b are real numbers such that $ab = 0$, then $a = 0$ or $b = 0$. However, if A and B are matrices such that $AB = O$, it is *not necessarily* true that $A = O$ or $B = O$. Illustrate this using the following matrices.

$$A = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

89. Finding Matrices Find two matrices A and B such that $AB = BA$.

90. HOW DO YOU SEE IT? A corporation has three factories that manufacture acoustic guitars and electric guitars. The production levels are represented by A.

$$\begin{array}{rc} & \overbrace{\begin{array}{ccc}\text{A} & \text{B} & \text{C}\end{array}}^{\text{Factory}} \\ A = & \begin{bmatrix} 70 & 50 & 25 \\ 35 & 100 & 70 \end{bmatrix} \end{array} \left.\begin{array}{l}\text{Acoustic}\\ \text{Electric}\end{array}\right\} \text{Guitar type}$$

(a) Interpret the value of a_{22}.

(b) How could you find the production levels when production increases by 20%?

(c) Each acoustic guitar sells for \$80 and each electric guitar sells for \$120. How could you use matrices to find the total sales value of the guitars produced at each factory?

91. Conjecture Let A and B be unequal diagonal matrices of the same dimension. (A **diagonal matrix** is a square matrix in which each entry not on the main diagonal is zero.) Determine the products AB for several pairs of such matrices. Make a conjecture about a rule that can be used to calculate AB without using row-by-column multiplication.

92. Matrices with Complex Entries Let $i = \sqrt{-1}$ and let

$$A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

(a) Find A^2, A^3, and A^4. Identify any similarities with i^2, i^3, and i^4.

(b) Find and identify B^2.

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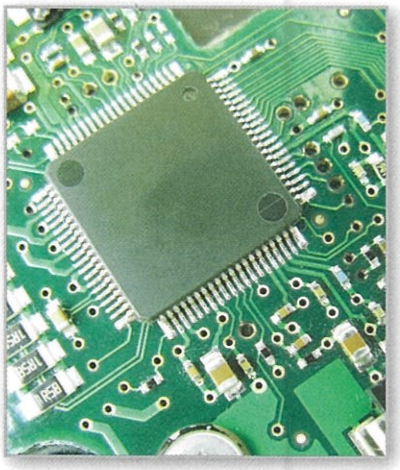
8.3 The Inverse of a Square Matrix



Inverse matrices are used to model and solve real-life problems. For example, in Exercises 59–62 on page 575, you will use an inverse matrix to find the currents in a circuit.

- **Verify that two matrices are inverses of each other.**
- **Use Gauss-Jordan elimination to find the inverses of matrices.**
- **Use a formula to find the inverses of 2 × 2 matrices.**
- **Use inverse matrices to solve systems of linear equations.**

The Inverse of a Matrix

This section further develops the algebra of matrices. To begin, consider the real number equation $ax = b$. To solve this equation for x, multiply each side of the equation by a^{-1} (provided that $a \neq 0$).

$$\begin{aligned} ax &= b \\ (a^{-1}a)x &= a^{-1}b \\ (1)x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

The number a^{-1} is called the *multiplicative inverse* of a because $a^{-1}a = 1$. The multiplicative **inverse of a matrix** is defined in a similar way.

Definition of the Inverse of a Square Matrix

Let A be an $n \times n$ matrix and let I_n be the $n \times n$ identity matrix. If there exists a matrix A^{-1} such that

$$AA^{-1} = I_n = A^{-1}A$$

then A^{-1} is the **inverse** of A. The symbol A^{-1} is read as "A inverse."

EXAMPLE 1 The Inverse of a Matrix

Show that $B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}$ is the inverse of $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$.

Solution To show that B is the inverse of A, show that $AB = I = BA$.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So, B is the inverse of A because $AB = I = BA$. This is an example of a square matrix that has an inverse. Note that not all square matrices have inverses.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Show that $B = \begin{bmatrix} -1 & -1 \\ -3 & -2 \end{bmatrix}$ is the inverse of $A = \begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$. ■

Recall that it is not always true that $AB = BA$, even when both products are defined. However, if A and B are both square matrices and $AB = I_n$, then it can be shown that $BA = I_n$. So, in Example 1, you need only to check that $AB = I_2$.

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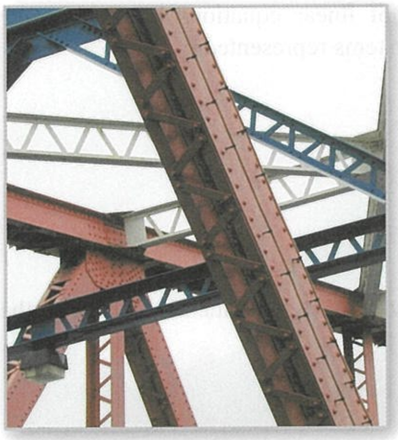

One real-life application of inverse matrices is in the study of beam deflection. In a simply supported elastic beam subjected to multiple forces, deflection **d** is related to force **w** by the matrix equation

$$\mathbf{d} = F\mathbf{w}$$

where F is a *flexibility matrix* whose entries depend on the material of the beam. The inverse of the flexibility matrix, F^{-1}, is the *stiffness matrix.*

Finding Inverse Matrices

If a matrix A has an inverse, then A is **invertible** (or **nonsingular**); otherwise, A is **singular.** A nonsquare matrix cannot have an inverse. To see this, note that when A is of dimension $m \times n$ and B is of dimension $n \times m$ (where $m \neq n$), the products AB and BA are of different dimensions and so cannot be equal to each other. Not all square matrices have inverses (see the matrix at the bottom of page 571). When a matrix does have an inverse, however, that inverse is unique. Example 2 shows how to use a system of equations to find the inverse of a matrix.

EXAMPLE 2 Finding the Inverse of a Matrix

Find the inverse of $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$.

Solution To find the inverse of A, solve the matrix equation $AX = I$ for X.

$$\overset{A}{\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}}\overset{X}{\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}} \overset{=}{=} \overset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \qquad \text{Write matrix equation.}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{Multiply.}$$

Equating corresponding entries, you obtain two systems of linear equations.

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases} \qquad \begin{cases} x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1 \end{cases}$$

Solve the first system using elementary row operations to determine that

$$x_{11} = -3 \quad \text{and} \quad x_{21} = 1.$$

Solve the second system to determine that

$$x_{12} = -4 \quad \text{and} \quad x_{22} = 1.$$

So, the inverse of A is

$$\begin{aligned} X &= A^{-1} \\ &= \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

Use matrix multiplication to check this result in two ways.

Check

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark \end{aligned}$$

$$\begin{aligned} A^{-1}A &= \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark \end{aligned}$$

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Find the inverse of $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$. ■

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In Example 2, note that the two systems of linear equations have the *same coefficient matrix A*. Rather than solve the two systems represented by

$$\left[\begin{array}{rrcr} 1 & 4 & \vdots & 1 \\ -1 & -3 & \vdots & 0 \end{array}\right]$$

and

$$\left[\begin{array}{rrcr} 1 & 4 & \vdots & 0 \\ -1 & -3 & \vdots & 1 \end{array}\right]$$

separately, you can solve them *simultaneously* by *adjoining* the identity matrix to the coefficient matrix to obtain

$$\begin{array}{c} \begin{array}{cc} \quad A \quad\quad\quad & \quad\quad I \end{array} \\ \left[\begin{array}{rrcrr} 1 & 4 & \vdots & 1 & 0 \\ -1 & -3 & \vdots & 0 & 1 \end{array}\right]. \end{array}$$

This "doubly augmented" matrix can be represented as

$$[A \ \vdots \ I].$$

By applying Gauss-Jordan elimination to this matrix, you can solve *both* systems with a single elimination process.

TECHNOLOGY Most graphing utilities can find the inverse of a square matrix. To do so, you may have to use the inverse key $\boxed{x^{-1}}$. Consult the user's guide for your graphing utility for specific keystrokes.

$$\begin{array}{rl} & \left[\begin{array}{rrcrr} 1 & 4 & \vdots & 1 & 0 \\ -1 & -3 & \vdots & 0 & 1 \end{array}\right] \\ R_1 + R_2 \rightarrow & \left[\begin{array}{rrcrr} 1 & 4 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & 1 & 1 \end{array}\right] \\ -4R_2 + R_1 \rightarrow & \left[\begin{array}{rrcrr} 1 & 0 & \vdots & -3 & -4 \\ 0 & 1 & \vdots & 1 & 1 \end{array}\right] \end{array}$$

So, from the "doubly augmented" matrix $[A \ \vdots \ I]$, you obtain the matrix $[I \ \vdots \ A^{-1}]$.

$$\begin{array}{c} \begin{array}{cc} \quad A \quad\quad\quad & \quad\quad I \end{array} \\ \left[\begin{array}{rrcrr} 1 & 4 & \vdots & 1 & 0 \\ -1 & -3 & \vdots & 0 & 1 \end{array}\right] \end{array} \Rightarrow \begin{array}{c} \begin{array}{cc} \quad I \quad\quad\quad & \quad\quad A^{-1} \end{array} \\ \left[\begin{array}{rrcrr} 1 & 0 & \vdots & -3 & -4 \\ 0 & 1 & \vdots & 1 & 1 \end{array}\right] \end{array}$$

This procedure (or algorithm) works for any square matrix that has an inverse.

Finding an Inverse Matrix

Let A be a square matrix of dimension $n \times n$.

1. Write the $n \times 2n$ matrix that consists of the given matrix A on the left and the $n \times n$ identity matrix I on the right to obtain

   $$[A \ \vdots \ I].$$

2. If possible, row reduce A to I using elementary row operations on the *entire* matrix

   $$[A \ \vdots \ I].$$

   The result will be the matrix

   $$[I \ \vdots \ A^{-1}].$$

   If this is not possible, then A is not invertible.

3. Check your work by multiplying to see that

   $$AA^{-1} = I = A^{-1}A.$$

EXAMPLE 3 Finding the Inverse of a Matrix

Find the inverse of

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix}.$$

Solution Begin by adjoining the identity matrix to A to form the matrix

$$[A \ \vdots \ I] = \begin{bmatrix} 1 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 1 & 0 & -1 & \vdots & 0 & 1 & 0 \\ 6 & -2 & -3 & \vdots & 0 & 0 & 1 \end{bmatrix}.$$

Use elementary row operations to obtain the form $[I \ \vdots \ A^{-1}]$.

$$\begin{array}{r} \\ -R_1 + R_2 \rightarrow \\ -6R_1 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & -1 & \vdots & -1 & 1 & 0 \\ 0 & 4 & -3 & \vdots & -6 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{r} R_2 + R_1 \rightarrow \\ \\ -4R_2 + R_3 \rightarrow \end{array} \begin{bmatrix} 1 & 0 & -1 & \vdots & 0 & 1 & 0 \\ 0 & 1 & -1 & \vdots & -1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & -2 & -4 & 1 \end{bmatrix}$$

$$\begin{array}{r} R_3 + R_1 \rightarrow \\ R_3 + R_2 \rightarrow \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & \vdots & -2 & -3 & 1 \\ 0 & 1 & 0 & \vdots & -3 & -3 & 1 \\ 0 & 0 & 1 & \vdots & -2 & -4 & 1 \end{bmatrix} = [I \ \vdots \ A^{-1}]$$

So, the matrix A is invertible and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}.$$

Check

$$AA^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

REMARK Be sure to check your solution because it is not uncommon to make arithmetic errors when using elementary row operations.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Find the inverse of

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 0 & -1 & 2 \\ 1 & -2 & 0 \end{bmatrix}.$$

The process shown in Example 3 applies to any $n \times n$ matrix A. When using this algorithm, if the matrix A does not reduce to the identity matrix, then A does not have an inverse. For example, the matrix below has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

To confirm that this matrix has no inverse, adjoin the identity matrix to A to form $[A \ \vdots \ I]$ and try to apply Gauss-Jordan elimination to the matrix. You will find that it is impossible to obtain the identity matrix I on the left. So, A is not invertible.

The Inverse of a 2 × 2 Matrix

Using Gauss-Jordan elimination to find the inverse of a matrix works well (even as a computer technique) for matrices of dimension 3 × 3 or greater. For 2 × 2 matrices, however, many people prefer to use a formula for the inverse rather than Gauss-Jordan elimination. This simple formula, which works *only* for 2 × 2 matrices, is explained as follows. A 2 × 2 matrix A given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if

$$ad - bc \neq 0.$$

Moreover, if $ad - bc \neq 0$, then the inverse is given by

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$ Formula for the inverse of a 2 × 2 matrix

The denominator

$$ad - bc$$

is the **determinant** of the 2 × 2 matrix A. You will study determinants in the next section.

EXAMPLE 4 Finding the Inverse of a 2 × 2 Matrix

See LarsonPrecalculus.com for an interactive version of this type of example.

If possible, find the inverse of each matrix.

a. $A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$ **b.** $B = \begin{bmatrix} 3 & -1 \\ -6 & 2 \end{bmatrix}$

Solution

a. The determinant of a matrix A is

$$ad - bc = 3(2) - (-1)(-2) = 4.$$

This quantity is not zero, so the matrix is invertible. The inverse is formed by interchanging the entries on the main diagonal, changing the signs of the other two entries, and multiplying by the scalar $\frac{1}{4}$.

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$ Formula for the inverse of a 2 × 2 matrix

$$= \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$$ Substitute for a, b, c, d, and the determinant.

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$ Multiply by the scalar $\frac{1}{4}$.

b. The determinant of matrix B is

$$ad - bc = 3(2) - (-1)(-6) = 0.$$

Because $ad - bc = 0$, B is not invertible.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

If possible, find the inverse of $A = \begin{bmatrix} 5 & -1 \\ 3 & 4 \end{bmatrix}$.

Systems of Linear Equations

You know that a system of linear equations can have exactly one solution, infinitely many solutions, or no solution. If the coefficient matrix A of a *square* system (a system that has the same number of equations as variables) is invertible, then the system has a unique solution, which can be found using an inverse matrix as follows.

A System of Equations with a Unique Solution

If A is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by $X = A^{-1}B$.

TECHNOLOGY On most graphing utilities, to solve a linear system that has an invertible coefficient matrix, you can use the formula $X = A^{-1}B$. That is, enter the $n \times n$ coefficient matrix $[A]$ and the $n \times 1$ column matrix $[B]$. The solution matrix X is given by

$$[A]^{-1}[B].$$

EXAMPLE 5 Solving a System Using an Inverse Matrix

Use an inverse matrix to solve the system

$$\begin{cases} x + y + z = 10{,}000 \\ 0.06x + 0.075y + 0.095z = 730. \\ x - 2z = 0 \end{cases}$$

Solution Begin by writing the system in the matrix form $AX = B$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0.06 & 0.075 & 0.095 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10{,}000 \\ 730 \\ 0 \end{bmatrix}$$

Then, use Gauss-Jordan elimination to find A^{-1}.

$$A^{-1} = \begin{bmatrix} 15 & -200 & -2 \\ -21.5 & 300 & 3.5 \\ 7.5 & -100 & -1.5 \end{bmatrix}$$

Finally, multiply B by A^{-1} on the left to obtain the solution.

$$X = A^{-1}B = \begin{bmatrix} 15 & -200 & -2 \\ -21.5 & 300 & 3.5 \\ 7.5 & -100 & -1.5 \end{bmatrix}\begin{bmatrix} 10{,}000 \\ 730 \\ 0 \end{bmatrix} = \begin{bmatrix} 4000 \\ 4000 \\ 2000 \end{bmatrix}$$

The solution of the system is $x = 4000$, $y = 4000$, and $z = 2000$, or $(4000, 4000, 2000)$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Use an inverse matrix to solve the system $\begin{cases} 2x + 3y + z = -1 \\ 3x + 3y + z = 1. \\ 2x + 4y + z = -2 \end{cases}$

Summarize (Section 8.3)

1. State the definition of the inverse of a square matrix *(page 568)*. For an example of how to show that a matrix is the inverse of another matrix, see Example 1.
2. Explain how to find an inverse matrix *(pages 569 and 570)*. For examples of finding inverse matrices, see Examples 2 and 3.
3. State the formula for the inverse of a 2×2 matrix *(page 572)*. For an example of using this formula to find an inverse matrix, see Example 4.
4. Explain how to use an inverse matrix to solve a system of linear equations *(page 573)*. For an example of using an inverse matrix to solve a system of linear equations, see Example 5.

8.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. If there exists an $n \times n$ matrix A^{-1} such that $AA^{-1} = I_n = A^{-1}A$, then A^{-1} is the ________ of A.
2. A matrix that has an inverse is invertible or ________. A matrix that does not have an inverse is ________.
3. A 2×2 matrix is invertible if and only if its ________ is not zero.
4. If A is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by $X =$ ________.

Skills and Applications

The Inverse of a Matrix **In Exercises 5–12, show that B is the inverse of A.**

5. $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

7. $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}, \quad B = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad B = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

9. $A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$

10. $A = \begin{bmatrix} -4 & 1 & 5 \\ -1 & 2 & 4 \\ 0 & -1 & -1 \end{bmatrix},$

$B = \frac{1}{4}\begin{bmatrix} -2 & 4 & 6 \\ 1 & -4 & -11 \\ -1 & 4 & 7 \end{bmatrix}$

11. $A = \begin{bmatrix} 2 & 0 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ -1 & 1 & -2 & 1 \\ 3 & -1 & 1 & 0 \end{bmatrix},$

$B = \frac{1}{3}\begin{bmatrix} -1 & 3 & -2 & -2 \\ -2 & 9 & -7 & -10 \\ 1 & 0 & -1 & -1 \\ 3 & -6 & 6 & 6 \end{bmatrix}$

12. $A = \begin{bmatrix} -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix},$

$B = \frac{1}{3}\begin{bmatrix} -3 & 1 & 1 & -3 \\ -3 & -1 & 2 & -3 \\ 0 & 1 & 1 & 0 \\ -3 & -2 & 1 & 0 \end{bmatrix}$

Finding the Inverse of a Matrix **In Exercises 13–24, find the inverse of the matrix, if possible.**

13. $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$

15. $\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

16. $\begin{bmatrix} -7 & 33 \\ 4 & -19 \end{bmatrix}$

17. $\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$

18. $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$

19. $\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

21. $\begin{bmatrix} -5 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 5 & 7 \end{bmatrix}$

22. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 5 & 5 \end{bmatrix}$

23. $\begin{bmatrix} -8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$

24. $\begin{bmatrix} 1 & 3 & -2 & 0 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Finding the Inverse of a Matrix **In Exercises 25–32, use the matrix capabilities of a graphing utility to find the inverse of the matrix, if possible.**

25. $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ -5 & -7 & -15 \end{bmatrix}$

26. $\begin{bmatrix} 10 & 5 & -7 \\ -5 & 1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$

27. $\begin{bmatrix} -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ 1 & 0 & -\frac{3}{2} \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$

28. $\begin{bmatrix} -\frac{5}{6} & \frac{1}{3} & \frac{11}{6} \\ 0 & \frac{2}{3} & 2 \\ 1 & -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}$

29. $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}$

30. $\begin{bmatrix} 0.6 & 0 & -0.3 \\ 0.7 & -1 & 0.2 \\ 1 & 0 & -0.9 \end{bmatrix}$

31. $\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

32. $\begin{bmatrix} 1 & -2 & -1 & -2 \\ 3 & -5 & -2 & -3 \\ 2 & -5 & -2 & -5 \\ -1 & 4 & 4 & 11 \end{bmatrix}$

Finding the Inverse of a 2 × 2 Matrix In Exercises 33–38, use the formula on page 572 to find the inverse of the 2 × 2 matrix, if possible.

33. $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ **34.** $\begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$

35. $\begin{bmatrix} -4 & -6 \\ 2 & 3 \end{bmatrix}$ **36.** $\begin{bmatrix} -12 & 3 \\ 5 & -2 \end{bmatrix}$

37. $\begin{bmatrix} 0.5 & 0.3 \\ 1.5 & 0.6 \end{bmatrix}$ **38.** $\begin{bmatrix} -1.25 & 0.625 \\ 0.16 & 0.32 \end{bmatrix}$

Solving a System Using an Inverse Matrix In Exercises 39–42, use the inverse matrix found in Exercise 15 to solve the system of linear equations.

39. $\begin{cases} x - 2y = 5 \\ 2x - 3y = 10 \end{cases}$ **40.** $\begin{cases} x - 2y = 0 \\ 2x - 3y = 3 \end{cases}$

41. $\begin{cases} x - 2y = 4 \\ 2x - 3y = 2 \end{cases}$ **42.** $\begin{cases} x - 2y = 1 \\ 2x - 3y = -2 \end{cases}$

Solving a System Using an Inverse Matrix In Exercises 43 and 44, use the inverse matrix found in Exercise 19 to solve the system of linear equations.

43. $\begin{cases} x + y + z = 0 \\ 3x + 5y + 4z = 5 \\ 3x + 6y + 5z = 2 \end{cases}$ **44.** $\begin{cases} x + y + z = -1 \\ 3x + 5y + 4z = 2 \\ 3x + 6y + 5z = 0 \end{cases}$

Solving a System Using an Inverse Matrix In Exercises 45 and 46, use the inverse matrix found in Exercise 32 to solve the system of linear equations.

45. $\begin{cases} x_1 - 2x_2 - x_3 - 2x_4 = 0 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 = 1 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 = -1 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 = 2 \end{cases}$

46. $\begin{cases} x_1 - 2x_2 - x_3 - 2x_4 = 1 \\ 3x_1 - 5x_2 - 2x_3 - 3x_4 = -2 \\ 2x_1 - 5x_2 - 2x_3 - 5x_4 = 0 \\ -x_1 + 4x_2 + 4x_3 + 11x_4 = -3 \end{cases}$

Solving a System Using an Inverse Matrix In Exercises 47–54, use an inverse matrix to solve the system of linear equations, if possible.

47. $\begin{cases} 5x + 4y = -1 \\ 2x + 5y = 3 \end{cases}$ **48.** $\begin{cases} 18x + 12y = 13 \\ 30x + 24y = 23 \end{cases}$

49. $\begin{cases} -0.4x + 0.8y = 1.6 \\ 2x - 4y = 5 \end{cases}$ **50.** $\begin{cases} 0.2x - 0.6y = 2.4 \\ -x + 1.4y = -8.8 \end{cases}$

51. $\begin{cases} 2.3x - 1.9y = 6 \\ 1.5x + 0.75y = -12 \end{cases}$ **52.** $\begin{cases} 5.1x - 3.4y = -20 \\ 0.9x - 0.6y = -51 \end{cases}$

53. $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases}$ **54.** $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

Using a Graphing Utility In Exercises 55 and 56, use the matrix capabilities of a graphing utility to solve the system of linear equations, if possible.

55. $\begin{cases} 5x - 3y + 2z = 2 \\ 2x + 2y - 3z = 3 \\ x - 7y + 7z = -4 \end{cases}$ **56.** $\begin{cases} 2x + 3y + 5z = 4 \\ 3x + 5y + 9z = 7 \\ 5x + 9y + 16z = 13 \end{cases}$

Investment Portfolio In Exercises 57 and 58, you invest in AAA-rated bonds, A-rated bonds, and B-rated bonds. The average yields are 4.5% on AAA bonds, 5% on A bonds, and 9% on B bonds. You invest twice as much in B bonds as in A bonds. Let x, y, and z represent the amounts invested in AAA, A, and B bonds, respectively.

$$\begin{cases} x + y + z = \text{(total investment)} \\ 0.045x + 0.05y + 0.09z = \text{(annual return)} \\ 2y - z = 0 \end{cases}$$

Use the inverse of the coefficient matrix of this system to find the amount invested in each type of bond for the given total investment and annual return.

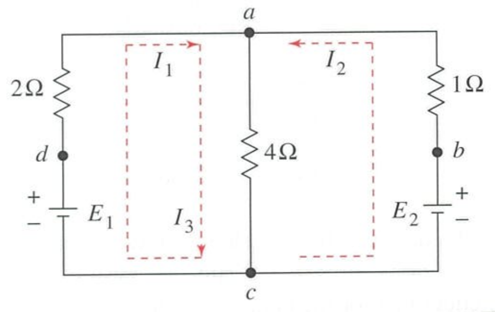
	Total Investment	Annual Return
57.	\$10,000	\$650
58.	\$12,000	\$835

Circuit Analysis

In Exercises 59–62, consider the circuit shown in the figure. The currents I_1, I_2, and I_3 (in amperes) are the solution of the system

$$\begin{cases} 2I_1 + 4I_3 = E_1 \\ I_2 + 4I_3 = E_2 \\ I_1 + I_2 - I_3 = 0 \end{cases}$$

where E_1 and E_2 are voltages. Use the inverse of the coefficient matrix of this system to find the unknown currents for the given voltages.

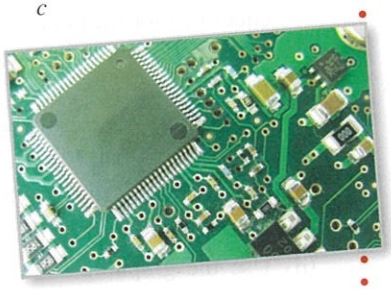


59. $E_1 = 15$ volts, $E_2 = 17$ volts

60. $E_1 = 10$ volts, $E_2 = 10$ volts

61. $E_1 = 28$ volts, $E_2 = 21$ volts

62. $E_1 = 24$ volts, $E_2 = 23$ volts



Ingrid Prats/Shutterstock.com

Raw Materials In Exercises 63 and 64, find the numbers of bags of potting soil that a company can produce for seedlings, general potting, and hardwood plants with the given amounts of raw materials. The raw materials used in one bag of each type of potting soil are shown below.

	Sand	Loam	Peat Moss
Seedlings	2 units	1 unit	1 unit
General	1 unit	2 units	1 unit
Hardwoods	2 units	2 units	2 units

63. 500 units of sand
500 units of loam
400 units of peat moss

64. 500 units of sand
750 units of loam
450 units of peat moss

65. Floral Design A florist is creating 10 centerpieces. Roses cost \$2.50 each, lilies cost \$4 each, and irises cost \$2 each. The customer has a budget of \$300 allocated for the centerpieces and wants each centerpiece to contain 12 flowers, with twice as many roses as the number of irises and lilies combined.

(a) Write a system of linear equations that represents the situation. Then write a matrix equation that corresponds to your system.

(b) Solve your system of linear equations using an inverse matrix. Find the number of flowers of each type that the florist can use to create the 10 centerpieces.

66. International Travel The table shows the numbers of visitors y (in thousands) to the United States from China from 2012 through 2014. *(Source: U.S. Department of Commerce)*

Year	Visitors, y (in thousands)
2012	1474
2013	1807
2014	2188

(a) The data can be modeled by the quadratic function $y = at^2 + bt + c$. Write a system of linear equations for the data. Let t represent the year, with $t = 12$ corresponding to 2012.

(b) Use the matrix capabilities of a graphing utility to find the inverse of the coefficient matrix of the system from part (a).

(c) Use the result of part (b) to solve the system and write the model $y = at^2 + bt + c$.

(d) Use the graphing utility to graph the model with the data.

Exploration

True or False? In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.

67. Multiplication of an invertible matrix and its inverse is commutative.

68. When the product of two square matrices is the identity matrix, the matrices are inverses of one another.

69. Writing Explain how to determine whether the inverse of a 2×2 matrix exists, as well as how to find the inverse when it exists.

70. Writing Explain how to write a system of three linear equations in three variables as a matrix equation $AX = B$, as well as how to solve the system using an inverse matrix.

Think About It In Exercises 71 and 72, find the value of k that makes the matrix singular.

71. $\begin{bmatrix} 4 & 3 \\ -2 & k \end{bmatrix}$

72. $\begin{bmatrix} 2k+1 & 3 \\ -7 & 1 \end{bmatrix}$

73. Conjecture Consider matrices of the form

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

(a) Write a 2×2 matrix and a 3×3 matrix in the form of A. Find the inverse of each.

(b) Use the result of part (a) to make a conjecture about the inverses of matrices in the form of A.

74. HOW DO YOU SEE IT? Consider the matrix

$$A = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}.$$

Use the determinant of A to state the conditions for which (a) A^{-1} exists and (b) $A^{-1} = A$.

75. Verifying a Formula Verify that the inverse of an invertible 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is given by $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Project: Consumer Credit To work an extended application analyzing the outstanding consumer credit in the United States, visit this text's website at *LarsonPrecalculus.com. (Source: Board of Governors of the Federal Reserve System)*