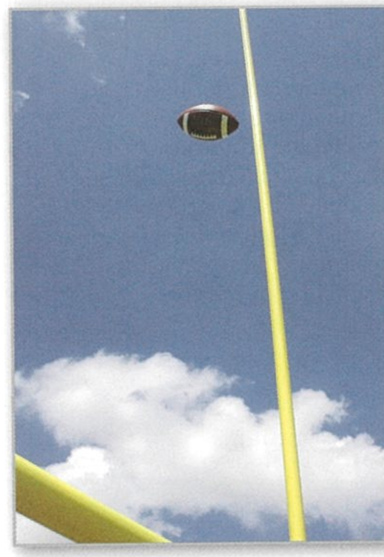
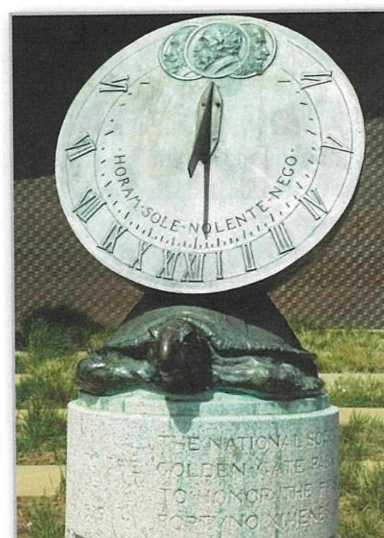
5 Analytic Trigonometry



Projectile Motion (Example 10, page 387)

Standing Waves (Exercise 80, page 379)

Ferris Wheel (Exercise 94, page 373)



Shadow Length (Exercise 62, page 361)

Friction (Exercise 65, page 354)

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5.1 Using Fundamental Identities

Fundamental trigonometric identities are useful in simplifying trigonometric expressions. For example, in Exercise 65 on page 354, you will use trigonometric identities to simplify an expression for the coefficient of friction.

- **Recognize and write the fundamental trigonometric identities.**
- **Use the fundamental trigonometric identities to evaluate trigonometric functions, simplify trigonometric expressions, and rewrite trigonometric expressions.**

Introduction

In Chapter 4, you studied the basic definitions, properties, graphs, and applications of the individual trigonometric functions. In this chapter, you will learn how to use the fundamental identities to perform the four tasks listed below.

1. Evaluate trigonometric functions.
2. Simplify trigonometric expressions.
3. Develop additional trigonometric identities.
4. Solve trigonometric equations.

Fundamental Trigonometric Identities

Reciprocal Identities

$\sin u = \dfrac{1}{\csc u}$ $\quad\cos u = \dfrac{1}{\sec u}$ $\quad\tan u = \dfrac{1}{\cot u}$

$\csc u = \dfrac{1}{\sin u}$ $\quad\sec u = \dfrac{1}{\cos u}$ $\quad\cot u = \dfrac{1}{\tan u}$

Quotient Identities

$\tan u = \dfrac{\sin u}{\cos u}$ $\quad\cot u = \dfrac{\cos u}{\sin u}$

Pythagorean Identities

$\sin^2 u + \cos^2 u = 1$ $\quad 1 + \tan^2 u = \sec^2 u$ $\quad 1 + \cot^2 u = \csc^2 u$

Cofunction Identities

$\sin\left(\dfrac{\pi}{2} - u\right) = \cos u$ $\quad\cos\left(\dfrac{\pi}{2} - u\right) = \sin u$

$\tan\left(\dfrac{\pi}{2} - u\right) = \cot u$ $\quad\cot\left(\dfrac{\pi}{2} - u\right) = \tan u$

$\sec\left(\dfrac{\pi}{2} - u\right) = \csc u$ $\quad\csc\left(\dfrac{\pi}{2} - u\right) = \sec u$

Even/Odd Identities

$\sin(-u) = -\sin u$ $\quad\cos(-u) = \cos u$ $\quad\tan(-u) = -\tan u$

$\csc(-u) = -\csc u$ $\quad\sec(-u) = \sec u$ $\quad\cot(-u) = -\cot u$

REMARK You should learn the fundamental trigonometric identities well, because you will use them frequently in trigonometry and they will also appear in calculus. Note that u can be an angle, a real number, or a variable.

Pythagorean identities are sometimes used in radical form such as

$$\sin u = \pm\sqrt{1 - \cos^2 u}$$

or

$$\tan u = \pm\sqrt{\sec^2 u - 1}$$

where the sign depends on the choice of u.

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Using the Fundamental Identities

One common application of trigonometric identities is to use given information about trigonometric functions to evaluate other trigonometric functions.

EXAMPLE 1 Using Identities to Evaluate a Function

Use the conditions $\sec u = -\frac{3}{2}$ and $\tan u > 0$ to find the values of all six trigonometric functions.

Solution Using a reciprocal identity, you have

$$\cos u = \frac{1}{\sec u} = \frac{1}{-3/2} = -\frac{2}{3}.$$

Using a Pythagorean identity, you have

$$\sin^2 u = 1 - \cos^2 u \qquad \text{Pythagorean identity}$$
$$= 1 - \left(-\tfrac{2}{3}\right)^2 \qquad \text{Substitute } -\tfrac{2}{3} \text{ for } \cos u.$$
$$= \tfrac{5}{9}. \qquad \text{Simplify.}$$

Because $\sec u < 0$ and $\tan u > 0$, it follows that u lies in Quadrant III. Moreover, $\sin u$ is negative when u is in Quadrant III, so choose the negative root and obtain $\sin u = -\sqrt{5}/3$. Knowing the values of the sine and cosine enables you to find the values of the remaining trigonometric functions.

$$\sin u = -\frac{\sqrt{5}}{3} \qquad \csc u = \frac{1}{\sin u} = -\frac{3}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$$

$$\cos u = -\frac{2}{3} \qquad \sec u = -\frac{3}{2}$$

$$\tan u = \frac{\sin u}{\cos u} = \frac{-\sqrt{5}/3}{-2/3} = \frac{\sqrt{5}}{2} \qquad \cot u = \frac{1}{\tan u} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Use the conditions $\tan x = \frac{1}{3}$ and $\cos x < 0$ to find the values of all six trigonometric functions.

EXAMPLE 2 Simplifying a Trigonometric Expression

Simplify the expression.

$$\sin x \cos^2 x - \sin x$$

Solution First factor out the common monomial factor $\sin x$ and then use a Pythagorean identity.

$$\sin x \cos^2 x - \sin x = \sin x(\cos^2 x - 1) \qquad \text{Factor out common monomial factor.}$$
$$= -\sin x(1 - \cos^2 x) \qquad \text{Factor out } -1.$$
$$= -\sin x(\sin^2 x) \qquad \text{Pythagorean identity}$$
$$= -\sin^3 x \qquad \text{Multiply.}$$

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Simplify the expression.

$$\cos^2 x \csc x - \csc x$$

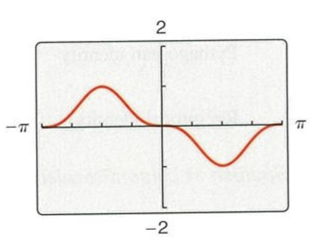
TECHNOLOGY Use a graphing utility to check the result of Example 2. To do this, enter

$$Y1 = -(\sin(X))^3$$

and

$$Y2 = \sin(X)(\cos(X))^2 - \sin(X).$$

Select the *line* style for Y1 and the *path* style for Y2, then graph both equations in the same viewing window. The two graphs *appear* to coincide, so it is reasonable to assume that their expressions are equivalent. Note that the actual equivalence of the expressions can only be verified algebraically, as in Example 2. This graphical approach is only to check your work.



When factoring trigonometric expressions, it is helpful to find a polynomial form that fits the expression, as shown in Example 3.

ALGEBRA HELP In Example 3, you factor the difference of two squares and you factor a trinomial. To review the techniques for factoring polynomials, see Appendix A.3.

EXAMPLE 3 Factoring Trigonometric Expressions

Factor each expression.

a. $\sec^2 \theta - 1$ **b.** $4 \tan^2 \theta + \tan \theta - 3$

Solution

a. This expression has the polynomial form $u^2 - v^2$, which is the difference of two squares. It factors as

$$\sec^2 \theta - 1 = (\sec \theta + 1)(\sec \theta - 1).$$

b. This expression has the polynomial form $ax^2 + bx + c$, and it factors as

$$4 \tan^2 \theta + \tan \theta - 3 = (4 \tan \theta - 3)(\tan \theta + 1).$$

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Factor each expression.

a. $1 - \cos^2 \theta$ **b.** $2 \csc^2 \theta - 7 \csc \theta + 6$

In some cases, when factoring or simplifying a trigonometric expression, it is helpful to first rewrite the expression in terms of just *one* trigonometric function or in terms of *sine and cosine only*. These strategies are demonstrated in Examples 4 and 5.

EXAMPLE 4 Factoring a Trigonometric Expression

Factor $\csc^2 x - \cot x - 3$.

Solution Use the identity $\csc^2 x = 1 + \cot^2 x$ to rewrite the expression.

$$\csc^2 x - \cot x - 3 = (1 + \cot^2 x) - \cot x - 3 \quad \text{Pythagorean identity}$$
$$= \cot^2 x - \cot x - 2 \quad \text{Combine like terms.}$$
$$= (\cot x - 2)(\cot x + 1) \quad \text{Factor.}$$

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Factor $\sec^2 x + 3 \tan x + 1$.

REMARK Remember that when adding rational expressions, you must first find the least common denominator (LCD). In Example 5, the LCD is $\sin t$.

EXAMPLE 5 Simplifying a Trigonometric Expression

See LarsonPrecalculus.com for an interactive version of this type of example.

$$\sin t + \cot t \cos t = \sin t + \left(\frac{\cos t}{\sin t}\right) \cos t \quad \text{Quotient identity}$$
$$= \frac{\sin^2 t + \cos^2 t}{\sin t} \quad \text{Add fractions.}$$
$$= \frac{1}{\sin t} \quad \text{Pythagorean identity}$$
$$= \csc t \quad \text{Reciprocal identity}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Simplify $\csc x - \cos x \cot x$.

EXAMPLE 6 Adding Trigonometric Expressions

Perform the addition and simplify: $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{\cos\theta}{\sin\theta}$.

Solution

$$\frac{\sin\theta}{1+\cos\theta}+\frac{\cos\theta}{\sin\theta}=\frac{(\sin\theta)(\sin\theta)+(\cos\theta)(1+\cos\theta)}{(1+\cos\theta)(\sin\theta)}$$

$$=\frac{\sin^2\theta+\cos^2\theta+\cos\theta}{(1+\cos\theta)(\sin\theta)}$$ Multiply.

$$=\frac{\cancel{1+\cos\theta}}{(\cancel{1+\cos\theta})(\sin\theta)}$$ Pythagorean identity

$$=\frac{1}{\sin\theta}$$ Divide out common factor.

$$=\csc\theta$$ Reciprocal identity

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Perform the addition and simplify: $\dfrac{1}{1+\sin\theta}+\dfrac{1}{1-\sin\theta}$.

The next two examples involve techniques for rewriting expressions in forms that are used in calculus.

EXAMPLE 7 Rewriting a Trigonometric Expression

Rewrite $\dfrac{1}{1+\sin x}$ so that it is *not* in fractional form.

Solution From the Pythagorean identity

$$\cos^2 x = 1-\sin^2 x=(1-\sin x)(1+\sin x)$$

multiplying both the numerator and the denominator by $(1-\sin x)$ will produce a monomial denominator.

$$\frac{1}{1+\sin x}=\frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x}$$ Multiply numerator and denominator by $(1-\sin x)$.

$$=\frac{1-\sin x}{1-\sin^2 x}$$ Multiply.

$$=\frac{1-\sin x}{\cos^2 x}$$ Pythagorean identity

$$=\frac{1}{\cos^2 x}-\frac{\sin x}{\cos^2 x}$$ Write as separate fractions.

$$=\frac{1}{\cos^2 x}-\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}$$ Product of fractions

$$=\sec^2 x-\tan x\sec x$$ Reciprocal and quotient identities

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Rewrite $\dfrac{\cos^2\theta}{1-\sin\theta}$ so that it is *not* in fractional form.

EXAMPLE 8 Trigonometric Substitution

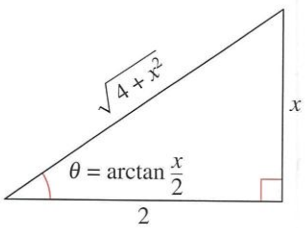
Use the substitution $x = 2 \tan \theta$, $0 < \theta < \pi/2$, to write $\sqrt{4 + x^2}$ as a trigonometric function of θ.

Solution Begin by letting $x = 2 \tan \theta$. Then, you obtain

$$\begin{aligned}
\sqrt{4 + x^2} &= \sqrt{4 + (2 \tan \theta)^2} && \text{Substitute } 2 \tan \theta \text{ for } x.\\
&= \sqrt{4 + 4 \tan^2 \theta} && \text{Property of exponents}\\
&= \sqrt{4(1 + \tan^2 \theta)} && \text{Factor.}\\
&= \sqrt{4 \sec^2 \theta} && \text{Pythagorean identity}\\
&= 2 \sec \theta. && \sec \theta > 0 \text{ for } 0 < \theta < \frac{\pi}{2}
\end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Use the substitution $x = 3 \sin \theta$, $0 < \theta < \pi/2$, to write $\sqrt{9 - x^2}$ as a trigonometric function of θ.



$2 \tan \theta = x \Rightarrow \tan \theta = \frac{x}{2}$

Figure 5.1

Figure 5.1 shows the right triangle illustration of the trigonometric substitution $x = 2 \tan \theta$ in Example 8. You can use this triangle to check the solution to Example 8. For $0 < \theta < \pi/2$, you have

$$\text{opp} = x, \quad \text{adj} = 2, \quad \text{and} \quad \text{hyp} = \sqrt{4 + x^2}.$$

Using these expressions,

$$\sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{4 + x^2}}{2}.$$

So, $2 \sec \theta = \sqrt{4 + x^2}$, and the solution checks.

EXAMPLE 9 Rewriting a Logarithmic Expression

Rewrite $\ln|\csc \theta| + \ln|\tan \theta|$ as a single logarithm and simplify the result.

Solution

$$\begin{aligned}
\ln|\csc \theta| + \ln|\tan \theta| &= \ln|\csc \theta \tan \theta| && \text{Product Property of Logarithms}\\
&= \ln\left|\frac{1}{\sin \theta} \cdot \frac{\sin \theta}{\cos \theta}\right| && \text{Reciprocal and quotient identities}\\
&= \ln\left|\frac{1}{\cos \theta}\right| && \text{Simplify.}\\
&= \ln|\sec \theta| && \text{Reciprocal identity}
\end{aligned}$$

▷ **ALGEBRA HELP** Recall that for positive real numbers u and v,

$$\ln u + \ln v = \ln(uv).$$

To review the properties of logarithms, see Section 3.3.

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Rewrite $\ln|\sec x| + \ln|\sin x|$ as a single logarithm and simplify the result.

Summarize (Section 5.1)

1. State the fundamental trigonometric identities *(page 348)*.
2. Explain how to use the fundamental trigonometric identities to evaluate trigonometric functions, simplify trigonometric expressions, and rewrite trigonometric expressions *(pages 349–352)*. For examples of these concepts, see Examples 1–9.

5.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blank to complete the trigonometric identity.

1. $\dfrac{\sin u}{\cos u} =$ ________
2. $\dfrac{1}{\sin u} =$ ________
3. $\dfrac{1}{\tan u} =$ ________
4. $\sec\left(\dfrac{\pi}{2} - u\right) =$ ________
5. $\sin^2 u + \cos^2 u =$ ________
6. $\sin(-u) =$ ________

Skills and Applications

Using Identities to Evaluate a Function In Exercises 7–12, use the given conditions to find the values of all six trigonometric functions.

7. $\sec x = -\frac{5}{2}$, $\tan x < 0$
8. $\csc x = -\frac{7}{6}$, $\tan x > 0$
9. $\sin \theta = -\frac{3}{4}$, $\cos \theta > 0$
10. $\cos \theta = \frac{2}{3}$, $\sin \theta < 0$
11. $\tan x = \frac{2}{3}$, $\cos x > 0$
12. $\cot x = \frac{7}{4}$, $\sin x < 0$

Matching Trigonometric Expressions In Exercises 13–18, match the trigonometric expression with its simplified form.

(a) $\csc x$ (b) -1 (c) 1

(d) $\sin x \tan x$ (e) $\sec^2 x$ (f) $\sec x$

13. $\sec x \cos x$
14. $\cot^2 x - \csc^2 x$
15. $\cos x(1 + \tan^2 x)$
16. $\cot x \sec x$
17. $\dfrac{\sec^2 x - 1}{\sin^2 x}$
18. $\dfrac{\cos^2[(\pi/2) - x]}{\cos x}$

Simplifying a Trigonometric Expression In Exercises 19–22, use the fundamental identities to simplify the expression. (There is more than one correct form of each answer).

19. $\dfrac{\tan \theta \cot \theta}{\sec \theta}$
20. $\cos\left(\dfrac{\pi}{2} - x\right) \sec x$
21. $\tan^2 x - \tan^2 x \sin^2 x$
22. $\sin^2 x \sec^2 x - \sin^2 x$

Factoring a Trigonometric Expression In Exercises 23–32, factor the expression. Use the fundamental identities to simplify, if necessary. (There is more than one correct form of each answer.)

23. $\dfrac{\sec^2 x - 1}{\sec x - 1}$
24. $\dfrac{\cos x - 2}{\cos^2 x - 4}$
25. $1 - 2\cos^2 x + \cos^4 x$
26. $\sec^4 x - \tan^4 x$
27. $\cot^3 x + \cot^2 x + \cot x + 1$
28. $\sec^3 x - \sec^2 x - \sec x + 1$
29. $3\sin^2 x - 5\sin x - 2$
30. $6\cos^2 x + 5\cos x - 6$
31. $\cot^2 x + \csc x - 1$
32. $\sin^2 x + 3\cos x + 3$

Simplifying a Trigonometric Expression In Exercises 33–40, use the fundamental identities to simplify the expression. (There is more than one correct form of each answer.)

33. $\tan \theta \csc \theta$
34. $\tan(-x) \cos x$
35. $\sin \phi(\csc \phi - \sin \phi)$
36. $\cos x(\sec x - \cos x)$
37. $\sin \beta \tan \beta + \cos \beta$
38. $\cot u \sin u + \tan u \cos u$
39. $\dfrac{1 - \sin^2 x}{\csc^2 x - 1}$
40. $\dfrac{\cos^2 y}{1 - \sin y}$

Multiplying Trigonometric Expressions In Exercises 41 and 42, perform the multiplication and use the fundamental identities to simplify. (There is more than one correct form of each answer.)

41. $(\sin x + \cos x)^2$
42. $(2\csc x + 2)(2\csc x - 2)$

Adding or Subtracting Trigonometric Expressions In Exercises 43–48, perform the addition or subtraction and use the fundamental identities to simplify. (There is more than one correct form of each answer.)

43. $\dfrac{1}{1 + \cos x} + \dfrac{1}{1 - \cos x}$
44. $\dfrac{1}{\sec x + 1} - \dfrac{1}{\sec x - 1}$
45. $\dfrac{\cos x}{1 + \sin x} - \dfrac{\cos x}{1 - \sin x}$
46. $\dfrac{\sin x}{1 + \cos x} + \dfrac{\sin x}{1 - \cos x}$
47. $\tan x - \dfrac{\sec^2 x}{\tan x}$
48. $\dfrac{\cos x}{1 + \sin x} + \dfrac{1 + \sin x}{\cos x}$

Rewriting a Trigonometric Expression In Exercises 49 and 50, rewrite the expression so that it is *not* in fractional form. (There is more than one correct form of each answer.)

49. $\dfrac{\sin^2 y}{1 - \cos y}$
50. $\dfrac{5}{\tan x + \sec x}$

Trigonometric Functions and Expressions In Exercises 51 and 52, use a graphing utility to determine which of the six trigonometric functions is equal to the expression. Verify your answer algebraically.

51. $\dfrac{\tan x + 1}{\sec x + \csc x}$ 52. $\dfrac{1}{\sin x}\left(\dfrac{1}{\cos x} - \cos x\right)$

Trigonometric Substitution In Exercises 53–56, use the trigonometric substitution to write the algebraic expression as a trigonometric function of θ, where $0 < \theta < \pi/2$.

53. $\sqrt{9 - x^2}$, $x = 3 \cos \theta$
54. $\sqrt{49 - x^2}$, $x = 7 \sin \theta$
55. $\sqrt{x^2 - 4}$, $x = 2 \sec \theta$
56. $\sqrt{9x^2 + 25}$, $3x = 5 \tan \theta$

Trigonometric Substitution In Exercises 57 and 58, use the trigonometric substitution to write the algebraic equation as a trigonometric equation of θ, where $-\pi/2 < \theta < \pi/2$. Then find $\sin \theta$ and $\cos \theta$.

57. $\sqrt{2} = \sqrt{4 - x^2}$, $x = 2 \sin \theta$
58. $5\sqrt{3} = \sqrt{100 - x^2}$, $x = 10 \cos \theta$

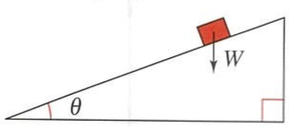
Solving a Trigonometric Equation In Exercises 59 and 60, use a graphing utility to solve the equation for θ, where $0 \le \theta < 2\pi$.

59. $\sin \theta = \sqrt{1 - \cos^2 \theta}$ 60. $\sec \theta = \sqrt{1 + \tan^2 \theta}$

Rewriting a Logarithmic Expression In Exercises 61–64, rewrite the expression as a single logarithm and simplify the result.

61. $\ln|\sin x| + \ln|\cot x|$ 62. $\ln|\cos x| - \ln|\sin x|$
63. $\ln|\tan t| - \ln(1 - \cos^2 t)$
64. $\ln(\cos^2 t) + \ln(1 + \tan^2 t)$

65. **Friction** The forces acting on an object weighing W units on an inclined plane positioned at an angle of θ with the horizontal (see figure) are modeled by $\mu W \cos \theta = W \sin \theta$, where μ is the coefficient of friction. Solve the equation for μ and simplify the result.



66. **Rate of Change** The rate of change of the function $f(x) = \sec x + \cos x$ is given by the expression $\sec x \tan x - \sin x$. Show that this expression can also be written as $\sin x \tan^2 x$.

Exploration

True or False? In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.

67. The quotient identities and reciprocal identities can be used to write any trigonometric function in terms of sine and cosine.

68. A cofunction identity can transform a tangent function into a cosecant function.

Analyzing Trigonometric Functions In Exercises 69 and 70, fill in the blanks. (*Note:* The notation $x \to c^+$ indicates that x approaches c from the right and $x \to c^-$ indicates that x approaches c from the left.)

69. As $x \to \left(\frac{\pi}{2}\right)^-$, $\tan x \to$ ▒ and $\cot x \to$ ▒.

70. As $x \to \pi^+$, $\sin x \to$ ▒ and $\csc x \to$ ▒.

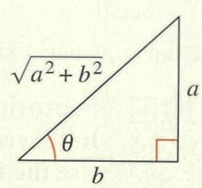
71. **Error Analysis** Describe the error.

$$\frac{\sin \theta}{\cos(-\theta)} = \frac{\sin \theta}{-\cos \theta}$$
$$= -\tan \theta$$

72. **Trigonometric Substitution** Use the trigonometric substitution $u = a \tan \theta$, where $-\pi/2 < \theta < \pi/2$ and $a > 0$, to simplify the expression $\sqrt{a^2 + u^2}$.

73. **Writing Trigonometric Functions in Terms of Sine** Write each of the other trigonometric functions of θ in terms of $\sin \theta$.

74. **HOW DO YOU SEE IT?**
Explain how to use the figure to derive the Pythagorean identities



$\sin^2 \theta + \cos^2 \theta = 1$,

$1 + \tan^2 \theta = \sec^2 \theta$,

and $1 + \cot^2 \theta = \csc^2 \theta$.

Discuss how to remember these identities and other fundamental trigonometric identities.

75. **Rewriting a Trigonometric Expression** Rewrite the expression below in terms of $\sin \theta$ and $\cos \theta$.

$$\frac{\sec \theta(1 + \tan \theta)}{\sec \theta + \csc \theta}$$

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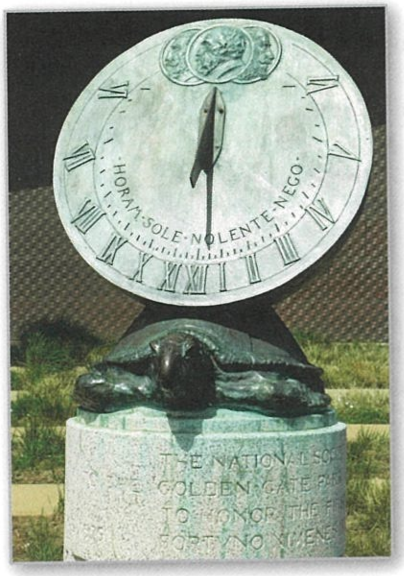
5.2 Verifying Trigonometric Identities



Trigonometric identities enable you to rewrite trigonometric equations that model real-life situations. For example, in Exercise 62 on page 361, trigonometric identities can help you simplify an equation that models the length of a shadow cast by a gnomon (a device used to tell time).

- **Verify trigonometric identities.**

Verifying Trigonometric Identities

In this section, you will study techniques for verifying trigonometric identities. In the next section, you will study techniques for solving trigonometric equations. The key to both verifying identities *and* solving equations is your ability to use the fundamental identities and the rules of algebra to rewrite trigonometric expressions.

Remember that a *conditional equation* is an equation that is true for only some of the values in the domain of the variable. For example, the conditional equation

$$\sin x = 0 \qquad \text{Conditional equation}$$

is true only for

$$x = n\pi$$

where n is an integer. When you are finding the values of the variable for which the equation is true, you are *solving* the equation.

On the other hand, an equation that is true for all real values in the domain of the variable is an *identity*. For example, the familiar equation

$$\sin^2 x = 1 - \cos^2 x \qquad \text{Identity}$$

is true for all real numbers x. So, it is an identity.

Although there are similarities, verifying that a trigonometric equation is an identity is quite different from solving an equation. There is no well-defined set of rules to follow in verifying trigonometric identities, the process is best learned through practice.

Guidelines for Verifying Trigonometric Identities

1. Work with one side of the equation at a time. It is often better to work with the more complicated side first.
2. Look for opportunities to factor an expression, add fractions, square a binomial, or create a monomial denominator.
3. Look for opportunities to use the fundamental identities. Note which functions are in the final expression you want. Sines and cosines pair up well, as do secants and tangents, and cosecants and cotangents.
4. When the preceding guidelines do not help, try converting all terms to sines and cosines.
5. Always try *something*. Even making an attempt that leads to a dead end can provide insight.

Verifying trigonometric identities is a useful process when you need to convert a trigonometric expression into a form that is more useful algebraically. When you verify an identity, you cannot *assume* that the two sides of the equation are equal because you are trying to verify that they *are* equal. As a result, when verifying identities, you cannot use operations such as adding the same quantity to each side of the equation or cross multiplication.

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EXAMPLE 1 Verifying a Trigonometric Identity

Verify the identity $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$.

REMARK Remember that an identity is only true for all real values in the domain of the variable. For instance, in Example 1 the identity is not true when $\theta = \pi/2$ because $\sec^2\theta$ is undefined when $\theta = \pi/2$.

Solution Start with the left side because it is more complicated.

$$\frac{\sec^2\theta - 1}{\sec^2\theta} = \frac{\tan^2\theta}{\sec^2\theta} \qquad \text{Pythagorean identity}$$

$$= \tan^2\theta(\cos^2\theta) \qquad \text{Reciprocal identity}$$

$$= \frac{\sin^2\theta}{\cancel{(\cos^2\theta)}}\cancel{(\cos^2\theta)} \qquad \text{Quotient identity}$$

$$= \sin^2\theta \qquad \text{Simplify.}$$

Notice that you verify the identity by starting with the left side of the equation (the more complicated side) and using the fundamental trigonometric identities to simplify it until you obtain the right side.

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Verify the identity $\dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sec^2\theta} = 1$.

There can be more than one way to verify an identity. Here is another way to verify the identity in Example 1.

$$\frac{\sec^2\theta - 1}{\sec^2\theta} = \frac{\sec^2\theta}{\sec^2\theta} - \frac{1}{\sec^2\theta} \qquad \text{Write as separate fractions.}$$

$$= 1 - \cos^2\theta \qquad \text{Reciprocal identity}$$

$$= \sin^2\theta \qquad \text{Pythagorean identity}$$

EXAMPLE 2 Verifying a Trigonometric Identity

Verify the identity $2\sec^2\alpha = \dfrac{1}{1 - \sin\alpha} + \dfrac{1}{1 + \sin\alpha}$.

Algebraic Solution

Start with the right side because it is more complicated.

$$\frac{1}{1 - \sin\alpha} + \frac{1}{1 + \sin\alpha} = \frac{1 + \sin\alpha + 1 - \sin\alpha}{(1 - \sin\alpha)(1 + \sin\alpha)} \qquad \text{Add fractions.}$$

$$= \frac{2}{1 - \sin^2\alpha} \qquad \text{Simplify.}$$

$$= \frac{2}{\cos^2\alpha} \qquad \text{Pythagorean identity}$$

$$= 2\sec^2\alpha \qquad \text{Reciprocal identity}$$

Numerical Solution

Use a graphing utility to create a table that shows the values of $y_1 = 2/\cos^2 x$ and $y_2 = [1/(1 - \sin x)] + [1/(1 + \sin x)]$ for different values of x.

X	Y1	Y2
-.5	2.5969	2.5969
-.25	2.1304	2.1304
0	2	2
.25	2.1304	2.1304
.5	2.5969	2.5969
.75	3.7357	3.7357
1	6.851	6.851

X=-.5

The values in the table for y_1 and y_2 appear to be identical, so the equation appears to be an identity.

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Verify the identity $2\csc^2\beta = \dfrac{1}{1 - \cos\beta} + \dfrac{1}{1 + \cos\beta}$.

EXAMPLE 3 Verifying a Trigonometric Identity

Verify the identity

$(\tan^2 x + 1)(\cos^2 x - 1) = -\tan^2 x.$

Algebraic Solution

Apply Pythagorean identities before multiplying.

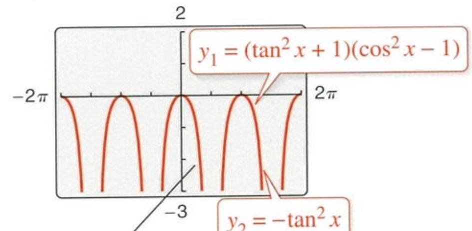
$$(\tan^2 x + 1)(\cos^2 x - 1) = (\sec^2 x)(-\sin^2 x) \quad \text{Pythagorean identities}$$

$$= -\frac{\sin^2 x}{\cos^2 x} \quad \text{Reciprocal identity}$$

$$= -\left(\frac{\sin x}{\cos x}\right)^2 \quad \text{Property of exponents}$$

$$= -\tan^2 x \quad \text{Quotient identity}$$

Graphical Solution



The graphs appear to coincide, so the given equation appears to be an identity.

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Verify the identity $(\sec^2 x - 1)(\sin^2 x - 1) = -\sin^2 x.$

EXAMPLE 4 Converting to Sines and Cosines

Verify each identity.

a. $\tan x \csc x = \sec x$

b. $\tan x + \cot x = \sec x \csc x$

Solution

a. Convert the left side into sines and cosines.

$$\tan x \csc x = \frac{\sin x}{\cos x} \cdot \frac{1}{\sin x} \quad \text{Quotient and reciprocal identities}$$

$$= \frac{1}{\cos x} \quad \text{Simplify.}$$

$$= \sec x \quad \text{Reciprocal identity}$$

b. Convert the left side into sines and cosines.

$$\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \quad \text{Quotient identities}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} \quad \text{Add fractions.}$$

$$= \frac{1}{\cos x \sin x} \quad \text{Pythagorean identity}$$

$$= \frac{1}{\cos x} \cdot \frac{1}{\sin x} \quad \text{Product of fractions}$$

$$= \sec x \csc x \quad \text{Reciprocal identities}$$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Verify each identity.

a. $\cot x \sec x = \csc x$

b. $\csc x - \sin x = \cos x \cot x$

ALGEBRA HELP To review the techniques for rationalizing a denominator, see Appendix A.2.

Recall from algebra that *rationalizing the denominator* using conjugates is, on occasion, a powerful simplification technique. A related form of this technique works for simplifying trigonometric expressions as well. For example, to simplify

$$\frac{1}{1 - \cos x}$$

multiply the numerator and the denominator by $1 + \cos x$.

$$\begin{aligned}\frac{1}{1 - \cos x} &= \frac{1}{1 - \cos x}\left(\frac{1 + \cos x}{1 + \cos x}\right)\\ &= \frac{1 + \cos x}{1 - \cos^2 x}\\ &= \frac{1 + \cos x}{\sin^2 x}\\ &= \csc^2 x(1 + \cos x)\end{aligned}$$

The expression $\csc^2 x(1 + \cos x)$ is considered a simplified form of

$$\frac{1}{1 - \cos x}$$

because $\csc^2 x(1 + \cos x)$ does not contain fractions.

EXAMPLE 5 Verifying a Trigonometric Identity

See LarsonPrecalculus.com for an interactive version of this type of example.

Verify the identity $\sec x + \tan x = \dfrac{\cos x}{1 - \sin x}$.

Algebraic Solution

Begin with the *right* side and create a monomial denominator by multiplying the numerator and the denominator by $1 + \sin x$.

$$\frac{\cos x}{1 - \sin x} = \frac{\cos x}{1 - \sin x}\left(\frac{1 + \sin x}{1 + \sin x}\right)$$ Multiply numerator and denominator by $1 + \sin x$.

$$= \frac{\cos x + \cos x \sin x}{1 - \sin^2 x}$$ Multiply.

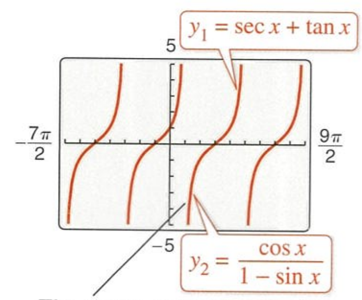
$$= \frac{\cos x + \cos x \sin x}{\cos^2 x}$$ Pythagorean identity

$$= \frac{\cos x}{\cos^2 x} + \frac{\cos x \sin x}{\cos^2 x}$$ Write as separate fractions.

$$= \frac{1}{\cos x} + \frac{\sin x}{\cos x}$$ Simplify.

$$= \sec x + \tan x$$ Identities

Graphical Solution



The graphs appear to coincide, so the given equation appears to be an identity.

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Verify the identity $\csc x + \cot x = \dfrac{\sin x}{1 - \cos x}$.

In Examples 1 through 5, you have been verifying trigonometric identities by working with one side of the equation and converting to the form given on the other side. On occasion, it is practical to work with each side *separately*, to obtain one common form that is equivalent to both sides. This is illustrated in Example 6.

EXAMPLE 6 Working with Each Side Separately

Verify the identity $\dfrac{\cot^2 \theta}{1 + \csc \theta} = \dfrac{1 - \sin \theta}{\sin \theta}$.

Algebraic Solution

Working with the left side, you have

$$\frac{\cot^2 \theta}{1 + \csc \theta} = \frac{\csc^2 \theta - 1}{1 + \csc \theta} \qquad \text{Pythagorean identity}$$

$$= \frac{(\csc \theta - 1)\cancel{(\csc \theta + 1)}}{\cancel{1 + \csc \theta}} \qquad \text{Factor.}$$

$$= \csc \theta - 1. \qquad \text{Simplify.}$$

Now, simplifying the right side, you have

$$\frac{1 - \sin \theta}{\sin \theta} = \frac{1}{\sin \theta} - \frac{\sin \theta}{\sin \theta} = \csc \theta - 1.$$

This verifies the identity because both sides are equal to $\csc \theta - 1$.

Numerical Solution

Use a graphing utility to create a table that shows the values of

$$y_1 = \frac{\cot^2 x}{1 + \csc x} \quad \text{and} \quad y_2 = \frac{1 - \sin x}{\sin x}$$

for different values of x.

X	Y1	Y2
-.5	-3.086	-3.086
-.25	-5.042	-5.042
0	ERROR	ERROR
.25	3.042	3.042
.5	1.0858	1.0858
.75	.46705	.46705
1	.1884	.1884
X=1		

The values for y_1 and y_2 appear to be identical, so the equation appears to be an identity.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Verify the identity $\dfrac{\tan^2 \theta}{1 + \sec \theta} = \dfrac{1 - \cos \theta}{\cos \theta}$.

Example 7 shows powers of trigonometric functions rewritten as more complicated sums of products of trigonometric functions. This is a common procedure used in calculus.

EXAMPLE 7 Two Examples from Calculus

Verify each identity.

a. $\tan^4 x = \tan^2 x \sec^2 x - \tan^2 x$ **b.** $\csc^4 x \cot x = \csc^2 x(\cot x + \cot^3 x)$

Solution

a. $\tan^4 x = (\tan^2 x)(\tan^2 x)$ Write as separate factors.

$= \tan^2 x(\sec^2 x - 1)$ Pythagorean identity

$= \tan^2 x \sec^2 x - \tan^2 x$ Multiply.

b. $\csc^4 x \cot x = \csc^2 x \csc^2 x \cot x$ Write as separate factors.

$= \csc^2 x(1 + \cot^2 x) \cot x$ Pythagorean identity

$= \csc^2 x(\cot x + \cot^3 x)$ Multiply.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Verify each identity.

a. $\tan^3 x = \tan x \sec^2 x - \tan x$ **b.** $\sin^3 x \cos^4 x = (\cos^4 x - \cos^6 x) \sin x$

Summarize (Section 5.2)

1. State the guidelines for verifying trigonometric identities *(page 355)*. For examples of verifying trigonometric identities, see Examples 1–7.

5.2 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary

In Exercises 1 and 2, fill in the blanks.

1. An equation that is true for all real values in the domain of the variable is an ________.
2. An equation that is true for only some values in the domain of the variable is a ________ ________.

In Exercises 3–8, fill in the blank to complete the fundamental trigonometric identity.

3. $\dfrac{1}{\cot u} = $ ______
4. $\dfrac{\cos u}{\sin u} = $ ______
5. $\cos\left(\dfrac{\pi}{2} - u\right) = $ ______
6. $1 + $ ______ $= \csc^2 u$
7. $\csc(-u) = $ ______
8. $\sec(-u) = $ ______

Skills and Applications

Verifying a Trigonometric Identity **In Exercises 9–18, verify the identity.**

9. $\tan t \cot t = 1$
10. $\dfrac{\tan x \cot x}{\cos x} = \sec x$
11. $(1 + \sin \alpha)(1 - \sin \alpha) = \cos^2 \alpha$
12. $\cos^2 \beta - \sin^2 \beta = 2\cos^2 \beta - 1$
13. $\cos^2 \beta - \sin^2 \beta = 1 - 2\sin^2 \beta$
14. $\sin^2 \alpha - \sin^4 \alpha = \cos^2 \alpha - \cos^4 \alpha$
15. $\tan\left(\dfrac{\pi}{2} - \theta\right) \tan \theta = 1$
16. $\dfrac{\cos[(\pi/2) - x]}{\sin[(\pi/2) - x]} = \tan x$
17. $\sin t \csc\left(\dfrac{\pi}{2} - t\right) = \tan t$
18. $\sec^2 y - \cot^2\left(\dfrac{\pi}{2} - y\right) = 1$

Verifying a Trigonometric Identity **In Exercises 19–24, verify the identity algebraically. Use the *table* feature of a graphing utility to check your result numerically.**

19. $\dfrac{1}{\tan x} + \dfrac{1}{\cot x} = \tan x + \cot x$
20. $\dfrac{1}{\sin x} - \dfrac{1}{\csc x} = \csc x - \sin x$
21. $\dfrac{1 + \sin \theta}{\cos \theta} + \dfrac{\cos \theta}{1 + \sin \theta} = 2 \sec \theta$
22. $\dfrac{\cos \theta \cot \theta}{1 - \sin \theta} - 1 = \csc \theta$
23. $\dfrac{1}{\cos x + 1} + \dfrac{1}{\cos x - 1} = -2 \csc x \cot x$
24. $\cos x - \dfrac{\cos x}{1 - \tan x} = \dfrac{\sin x \cos x}{\sin x - \cos x}$

Verifying a Trigonometric Identity **In Exercises 25–30, verify the identity algebraically. Use a graphing utility to check your result graphically.**

25. $\sec y \cos y = 1$
26. $\cot^2 y(\sec^2 y - 1) = 1$
27. $\dfrac{\tan^2 \theta}{\sec \theta} = \sin \theta \tan \theta$
28. $\dfrac{\cot^3 t}{\csc t} = \cos t(\csc^2 t - 1)$
29. $\dfrac{1}{\tan \beta} + \tan \beta = \dfrac{\sec^2 \beta}{\tan \beta}$
30. $\dfrac{\sec \theta - 1}{1 - \cos \theta} = \sec \theta$

Converting to Sines and Cosines **In Exercises 31–36, verify the identity by converting the left side into sines and cosines.**

31. $\dfrac{\cot^2 t}{\csc t} = \dfrac{1 - \sin^2 t}{\sin t}$
32. $\cos x + \sin x \tan x = \sec x$
33. $\sec x - \cos x = \sin x \tan x$
34. $\cot x - \tan x = \sec x(\csc x - 2 \sin x)$
35. $\dfrac{\cot x}{\sec x} = \csc x - \sin x$
36. $\dfrac{\csc(-x)}{\sec(-x)} = -\cot x$

Verifying a Trigonometric Identity **In Exercises 37–42, verify the identity.**

37. $\sin^{1/2} x \cos x - \sin^{5/2} x \cos x = \cos^3 x \sqrt{\sin x}$
38. $\sec^6 x(\sec x \tan x) - \sec^4 x(\sec x \tan x) = \sec^5 x \tan^3 x$
39. $(1 + \sin y)[1 + \sin(-y)] = \cos^2 y$
40. $\dfrac{\tan x + \tan y}{1 - \tan x \tan y} = \dfrac{\cot x + \cot y}{\cot x \cot y - 1}$
41. $\sqrt{\dfrac{1 + \sin \theta}{1 - \sin \theta}} = \dfrac{1 + \sin \theta}{|\cos \theta|}$
42. $\dfrac{\cos x - \cos y}{\sin x + \sin y} + \dfrac{\sin x - \sin y}{\cos x + \cos y} = 0$

Error Analysis **In Exercises 43 and 44, describe the error(s).**

43. $\dfrac{1}{\tan x} + \cot(-x) = \cot x + \cot x = 2\cot x$

44. $\dfrac{1+\sec(-\theta)}{\sin(-\theta)+\tan(-\theta)} = \dfrac{1-\sec\theta}{\sin\theta-\tan\theta}$

$= \dfrac{1-\sec\theta}{(\sin\theta)[1-(1/\cos\theta)]}$

$= \dfrac{1-\sec\theta}{\sin\theta(1-\sec\theta)}$

$= \dfrac{1}{\sin\theta}$

$= \csc\theta$

Determining Trigonometric Identities **In Exercises 45–50, (a) use a graphing utility to graph each side of the equation to determine whether the equation is an identity, (b) use the *table* feature of the graphing utility to determine whether the equation is an identity, and (c) confirm the results of parts (a) and (b) algebraically.**

45. $(1+\cot^2 x)(\cos^2 x) = \cot^2 x$

46. $\csc x(\csc x - \sin x) + \dfrac{\sin x - \cos x}{\sin x} + \cot x = \csc^2 x$

47. $2 + \cos^2 x - 3\cos^4 x = \sin^2 x(3 + 2\cos^2 x)$

48. $\tan^4 x + \tan^2 x - 3 = \sec^2 x(4\tan^2 x - 3)$

49. $\dfrac{1+\cos x}{\sin x} = \dfrac{\sin x}{1-\cos x}$

50. $\dfrac{\cot\alpha}{\csc\alpha+1} = \dfrac{\csc\alpha+1}{\cot\alpha}$

Verifying a Trigonometric Identity **In Exercises 51–54, verify the identity.**

51. $\tan^5 x = \tan^3 x\sec^2 x - \tan^3 x$

52. $\sec^4 x\tan^2 x = (\tan^2 x + \tan^4 x)\sec^2 x$

53. $\cos^3 x\sin^2 x = (\sin^2 x - \sin^4 x)\cos x$

54. $\sin^4 x + \cos^4 x = 1 - 2\cos^2 x + 2\cos^4 x$

Using Cofunction Identities **In Exercises 55 and 56, use the cofunction identities to evaluate the expression without using a calculator.**

55. $\sin^2 25^\circ + \sin^2 65^\circ$

56. $\tan^2 63^\circ + \cot^2 16^\circ - \sec^2 74^\circ - \csc^2 27^\circ$

Verifying a Trigonometric Identity **In Exercises 57–60, verify the identity.**

57. $\tan(\sin^{-1} x) = \dfrac{x}{\sqrt{1-x^2}}$

58. $\cos(\sin^{-1} x) = \sqrt{1-x^2}$

59. $\tan\left(\sin^{-1}\dfrac{x-1}{4}\right) = \dfrac{x-1}{\sqrt{16-(x-1)^2}}$

60. $\tan\left(\cos^{-1}\dfrac{x+1}{2}\right) = \dfrac{\sqrt{4-(x+1)^2}}{x+1}$

61. **Rate of Change** The rate of change of the function $f(x) = \sin x + \csc x$ is given by the expression $\cos x - \csc x\cot x$. Show that the expression for the rate of change can also be written as $-\cos x\cot^2 x$.

62. **Shadow Length**

The length s of a shadow cast by a vertical gnomon (a device used to tell time) of height h when the angle of the sun above the horizon is θ can be modeled by the equation

$$s = \frac{h\sin(90^\circ - \theta)}{\sin\theta},$$

$0^\circ < \theta \le 90^\circ$.

(a) Verify that the expression for s is equal to $h\cot\theta$.

(b) Use a graphing utility to create a table of the lengths s for different values of θ. Let $h = 5$ feet.

(c) Use your table from part (b) to determine the angle of the sun that results in the minimum length of the shadow.

(d) Based on your results from part (c), what time of day do you think it is when the angle of the sun above the horizon is 90°?

Exploration

True or False? **In Exercises 63–65, determine whether the statement is true or false. Justify your answer.**

63. $\tan x^2 = \tan^2 x$

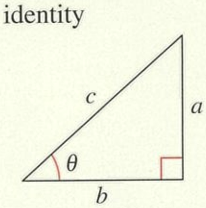
64. $\cos\left(\theta - \dfrac{\pi}{2}\right) = \sin\theta$

65. The equation $\sin^2\theta + \cos^2\theta = 1 + \tan^2\theta$ is an identity because $\sin^2(0) + \cos^2(0) = 1$ and $1 + \tan^2(0) = 1$.

66. **HOW DO YOU SEE IT?** Explain how to use the figure to derive the identity

$$\frac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$$

given in Example 1.



Think About It **In Exercises 67–70, explain why the equation is not an identity and find one value of the variable for which the equation is not true.**

67. $\sin\theta = \sqrt{1-\cos^2\theta}$

68. $\tan\theta = \sqrt{\sec^2\theta - 1}$

69. $1 - \cos\theta = \sin\theta$

70. $1 + \tan\theta = \sec\theta$

5.3 Solving Trigonometric Equations

Trigonometric equations have many applications in circular motion. For example, in Exercise 94 on page 373, you will solve a trigonometric equation to determine when a person riding a Ferris wheel will be at certain heights above the ground.

- **Use standard algebraic techniques to solve trigonometric equations.**
- **Solve trigonometric equations of quadratic type.**
- **Solve trigonometric equations involving multiple angles.**
- **Use inverse trigonometric functions to solve trigonometric equations.**

Introduction

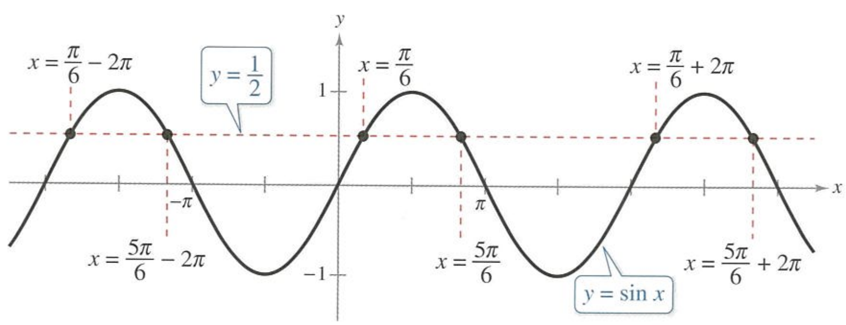
To solve a trigonometric equation, use standard algebraic techniques (when possible) such as collecting like terms, extracting square roots, and factoring. Your preliminary goal in solving a trigonometric equation is to *isolate* the trigonometric function on one side of the equation. For example, to solve the equation $2 \sin x = 1$, divide each side by 2 to obtain

$$\sin x = \frac{1}{2}.$$

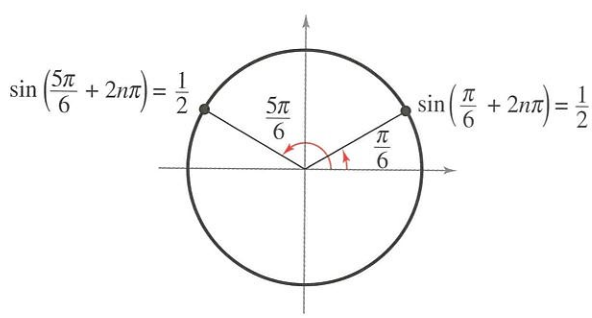
To solve for x, note in the graph of $y = \sin x$ below that the equation $\sin x = \frac{1}{2}$ has solutions $x = \pi/6$ and $x = 5\pi/6$ in the interval $[0, 2\pi)$. Moreover, because $\sin x$ has a period of 2π, there are infinitely many other solutions, which can be written as

$$x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + 2n\pi \qquad \text{General solution}$$

where n is an integer. Notice the solutions for $n = \pm 1$ in the graph of $y = \sin x$.



The figure below illustrates another way to show that the equation $\sin x = \frac{1}{2}$ has infinitely many solutions. Any angles that are coterminal with $\pi/6$ or $5\pi/6$ are also solutions of the equation.



When solving trigonometric equations, write your answer(s) using exact values (when possible) rather than decimal approximations.

iStockphoto.com/Flory

EXAMPLE 1 Collecting Like Terms

Solve

$$\sin x + \sqrt{2} = -\sin x.$$

Solution Begin by isolating $\sin x$ on one side of the equation.

$\sin x + \sqrt{2} = -\sin x$	Write original equation.
$\sin x + \sin x + \sqrt{2} = 0$	Add $\sin x$ to each side.
$\sin x + \sin x = -\sqrt{2}$	Subtract $\sqrt{2}$ from each side.
$2 \sin x = -\sqrt{2}$	Combine like terms.
$\sin x = -\frac{\sqrt{2}}{2}$	Divide each side by 2.

The period of $\sin x$ is 2π, so first find all solutions in the interval $[0, 2\pi)$. These solutions are $x = 5\pi/4$ and $x = 7\pi/4$. Finally, add multiples of 2π to each of these solutions to obtain the general form

$$x = \frac{5\pi}{4} + 2n\pi \quad \text{and} \quad x = \frac{7\pi}{4} + 2n\pi \qquad \text{General solution}$$

where n is an integer.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Solve $\sin x - \sqrt{2} = -\sin x$.

EXAMPLE 2 Extracting Square Roots

Solve

$$3 \tan^2 x - 1 = 0.$$

Solution Begin by isolating $\tan x$ on one side of the equation.

$3 \tan^2 x - 1 = 0$	Write original equation.
$3 \tan^2 x = 1$	Add 1 to each side.
$\tan^2 x = \frac{1}{3}$	Divide each side by 3.
$\tan x = \pm\frac{1}{\sqrt{3}}$	Extract square roots.
$\tan x = \pm\frac{\sqrt{3}}{3}$	Rationalize the denominator.

REMARK When you extract square roots, make sure you account for both the positive and negative solutions.

The period of $\tan x$ is π, so first find all solutions in the interval $[0, \pi)$. These solutions are $x = \pi/6$ and $x = 5\pi/6$. Finally, add multiples of π to each of these solutions to obtain the general form

$$x = \frac{\pi}{6} + n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + n\pi \qquad \text{General solution}$$

where n is an integer.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Solve $4 \sin^2 x - 3 = 0$.

The equations in Examples 1 and 2 involved only one trigonometric function. When two or more functions occur in the same equation, collect all terms on one side and try to separate the functions by factoring or by using appropriate identities. This may produce factors that yield no solutions, as illustrated in Example 3.

EXAMPLE 3 Factoring

Solve $\cot x \cos^2 x = 2 \cot x$.

Solution Begin by collecting all terms on one side of the equation and factoring.

$$\cot x \cos^2 x = 2 \cot x \qquad \text{Write original equation.}$$

$$\cot x \cos^2 x - 2 \cot x = 0 \qquad \text{Subtract } 2 \cot x \text{ from each side.}$$

$$\cot x(\cos^2 x - 2) = 0 \qquad \text{Factor.}$$

Set each factor equal to zero and isolate the trigonometric function, if necessary.

$$\cot x = 0 \quad \text{or} \quad \cos^2 x - 2 = 0$$

$$\cos^2 x = 2$$

$$\cos x = \pm\sqrt{2}$$

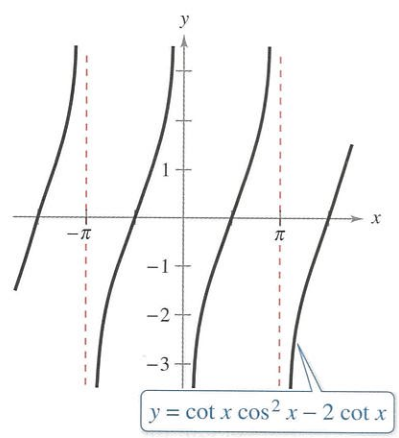
In the interval $(0, \pi)$, the equation $\cot x = 0$ has the solution

$$x = \frac{\pi}{2}.$$

No solution exists for $\cos x = \pm\sqrt{2}$ because $\pm\sqrt{2}$ are outside the range of the cosine function. The period of $\cot x$ is π, so add multiples of π to $x = \pi/2$ to get the general form

$$x = \frac{\pi}{2} + n\pi \qquad \text{General solution}$$

where n is an integer. Confirm this graphically by sketching the graph of $y = \cot x \cos^2 x - 2 \cot x$.



Notice that the x-intercepts occur at

$$-\frac{3\pi}{2}, \quad -\frac{\pi}{2}, \quad \frac{\pi}{2}, \quad \frac{3\pi}{2}$$

and so on. These x-intercepts correspond to the solutions of $\cot x \cos^2 x = 2 \cot x$.

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Solve $\sin^2 x = 2 \sin x$.

▷ **ALGEBRA HELP** To review the techniques for solving quadratic equations, see Appendix A.5.

Equations of Quadratic Type

Below are two examples of trigonometric equations of quadratic type

$ax^2 + bx + c = 0.$

To solve equations of this type, use factoring (when possible) or use the Quadratic Formula.

Quadratic in sin x	**Quadratic in sec x**
$2\sin^2 x - \sin x - 1 = 0$	$\sec^2 x - 3\sec x - 2 = 0$
$2(\sin x)^2 - (\sin x) - 1 = 0$	$(\sec x)^2 - 3(\sec x) - 2 = 0$

EXAMPLE 4 Solving an Equation of Quadratic Type

Find all solutions of $2\sin^2 x - \sin x - 1 = 0$ in the interval $[0, 2\pi)$.

Algebraic Solution

Treat the equation as quadratic in $\sin x$ and factor.

$2\sin^2 x - \sin x - 1 = 0$ Write original equation.

$(2\sin x + 1)(\sin x - 1) = 0$ Factor.

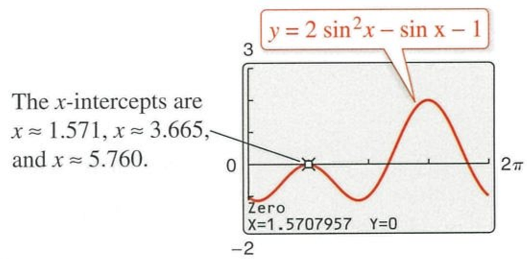
Setting each factor equal to zero, you obtain the following solutions in the interval $[0, 2\pi)$.

$2\sin x + 1 = 0$ or $\sin x - 1 = 0$

$\sin x = -\frac{1}{2}$ $\sin x = 1$

$x = \frac{7\pi}{6}, \frac{11\pi}{6}$ $x = \frac{\pi}{2}$

Graphical Solution



Use the x-intercepts to conclude that the approximate solutions of $2\sin^2 x - \sin x - 1 = 0$ in the interval $[0, 2\pi)$ are

$x \approx 1.571 \approx \frac{\pi}{2}$, $x \approx 3.665 \approx \frac{7\pi}{6}$, and $x \approx 5.760 \approx \frac{11\pi}{6}$.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Find all solutions of $2\sin^2 x - 3\sin x + 1 = 0$ in the interval $[0, 2\pi)$.

EXAMPLE 5 Rewriting with a Single Trigonometric Function

Solve $2\sin^2 x + 3\cos x - 3 = 0$.

Solution This equation contains both sine and cosine functions. Rewrite the equation so that it has only cosine functions by using the identity $\sin^2 x = 1 - \cos^2 x$.

$2\sin^2 x + 3\cos x - 3 = 0$ Write original equation.

$2(1 - \cos^2 x) + 3\cos x - 3 = 0$ Pythagorean identity

$2\cos^2 x - 3\cos x + 1 = 0$ Multiply each side by -1.

$(2\cos x - 1)(\cos x - 1) = 0$ Factor.

Setting each factor equal to zero, you obtain the solutions $x = 0$, $x = \pi/3$, and $x = 5\pi/3$ in the interval $[0, 2\pi)$. Because $\cos x$ has a period of 2π, the general solution is

$x = 2n\pi$, $x = \frac{\pi}{3} + 2n\pi$, and $x = \frac{5\pi}{3} + 2n\pi$ General solution

where n is an integer.

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Solve $3\sec^2 x - 2\tan^2 x - 4 = 0$.

Sometimes you square each side of an equation to obtain an equation of quadratic type, as demonstrated in the next example. This procedure can introduce extraneous solutions, so check any solutions in the original equation to determine whether they are valid or extraneous.

REMARK You square each side of the equation in Example 6 because the squares of the sine and cosine functions are related by a Pythagorean identity. The same is true for the squares of the secant and tangent functions and for the squares of the cosecant and cotangent functions.

EXAMPLE 6 Squaring and Converting to Quadratic Type

See LarsonPrecalculus.com for an interactive version of this type of example.

Find all solutions of $\cos x + 1 = \sin x$ in the interval $[0, 2\pi)$.

Solution It is not clear how to rewrite this equation in terms of a single trigonometric function. Notice what happens when you square each side of the equation.

$$\cos x + 1 = \sin x \quad \text{Write original equation.}$$

$$\cos^2 x + 2\cos x + 1 = \sin^2 x \quad \text{Square each side.}$$

$$\cos^2 x + 2\cos x + 1 = 1 - \cos^2 x \quad \text{Pythagorean identity}$$

$$\cos^2 x + \cos^2 x + 2\cos x + 1 - 1 = 0 \quad \text{Rewrite equation.}$$

$$2\cos^2 x + 2\cos x = 0 \quad \text{Combine like terms.}$$

$$2\cos x(\cos x + 1) = 0 \quad \text{Factor.}$$

Set each factor equal to zero and solve for x.

$$2\cos x = 0 \qquad \text{or} \qquad \cos x + 1 = 0$$

$$\cos x = 0 \qquad\qquad \cos x = -1$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad\qquad x = \pi$$

Because you squared the original equation, check for extraneous solutions.

Check $x = \frac{\pi}{2}$

$$\cos\frac{\pi}{2} + 1 \stackrel{?}{=} \sin\frac{\pi}{2} \quad \text{Substitute } \frac{\pi}{2} \text{ for } x.$$

$$0 + 1 = 1 \quad \text{Solution checks. } \checkmark$$

Check $x = \frac{3\pi}{2}$

$$\cos\frac{3\pi}{2} + 1 \stackrel{?}{=} \sin\frac{3\pi}{2} \quad \text{Substitute } \frac{3\pi}{2} \text{ for } x.$$

$$0 + 1 \neq -1 \quad \text{Solution does not check.}$$

Check $x = \pi$

$$\cos\pi + 1 \stackrel{?}{=} \sin\pi \quad \text{Substitute } \pi \text{ for } x.$$

$$-1 + 1 = 0 \quad \text{Solution checks. } \checkmark$$

Of the three possible solutions, $x = 3\pi/2$ is extraneous. So, in the interval $[0, 2\pi)$, the only two solutions are

$$x = \frac{\pi}{2} \quad \text{and} \quad x = \pi.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Find all solutions of $\sin x + 1 = \cos x$ in the interval $[0, 2\pi)$.

Functions Involving Multiple Angles

The next two examples involve trigonometric functions of multiple angles of the forms cos ku and tan ku. To solve equations involving these forms, first solve the equation for ku, and then divide your result by k.

EXAMPLE 7 Solving a Multiple-Angle Equation

Solve $2\cos 3t - 1 = 0$.

Solution

$$2\cos 3t - 1 = 0 \qquad \text{Write original equation.}$$

$$2\cos 3t = 1 \qquad \text{Add 1 to each side.}$$

$$\cos 3t = \frac{1}{2} \qquad \text{Divide each side by 2.}$$

In the interval $[0, 2\pi)$, you know that $3t = \pi/3$ and $3t = 5\pi/3$ are the only solutions, so, in general, you have

$$3t = \frac{\pi}{3} + 2n\pi \quad \text{and} \quad 3t = \frac{5\pi}{3} + 2n\pi.$$

Dividing these results by 3, you obtain the general solution

$$t = \frac{\pi}{9} + \frac{2n\pi}{3} \quad \text{and} \quad t = \frac{5\pi}{9} + \frac{2n\pi}{3} \qquad \text{General solution}$$

where n is an integer.

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Solve $2\sin 2t - \sqrt{3} = 0$.

EXAMPLE 8 Solving a Multiple-Angle Equation

$$3\tan\frac{x}{2} + 3 = 0 \qquad \text{Original equation}$$

$$3\tan\frac{x}{2} = -3 \qquad \text{Subtract 3 from each side.}$$

$$\tan\frac{x}{2} = -1 \qquad \text{Divide each side by 3.}$$

In the interval $[0, \pi)$, you know that $x/2 = 3\pi/4$ is the only solution, so, in general, you have

$$\frac{x}{2} = \frac{3\pi}{4} + n\pi.$$

Multiplying this result by 2, you obtain the general solution

$$x = \frac{3\pi}{2} + 2n\pi \qquad \text{General solution}$$

where n is an integer.

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Solve $2\tan\frac{x}{2} - 2 = 0$.

Using Inverse Functions

EXAMPLE 9 **Using Inverse Functions**

$$\sec^2 x - 2\tan x = 4 \quad \text{Original equation}$$

$$1 + \tan^2 x - 2\tan x - 4 = 0 \quad \text{Pythagorean identity}$$

$$\tan^2 x - 2\tan x - 3 = 0 \quad \text{Combine like terms.}$$

$$(\tan x - 3)(\tan x + 1) = 0 \quad \text{Factor.}$$

Setting each factor equal to zero, you obtain two solutions in the interval $(-\pi/2, \pi/2)$. [Recall that the range of the inverse tangent function is $(-\pi/2, \pi/2)$.]

$$x = \arctan 3 \quad \text{and} \quad x = \arctan(-1) = -\pi/4$$

Finally, $\tan x$ has a period of π, so add multiples of π to obtain

$$x = \arctan 3 + n\pi \quad \text{and} \quad x = (-\pi/4) + n\pi \quad \text{General solution}$$

where n is an integer. You can use a calculator to approximate the value of arctan 3.

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Solve $4\tan^2 x + 5\tan x - 6 = 0$.

EXAMPLE 10 **Using the Quadratic Formula**

Find all solutions of $\sin^2 x - 3\sin x - 2 = 0$ in the interval $[0, 2\pi)$.

Solution

The expression $\sin^2 x - 3\sin x - 2$ cannot be factored, so use the Quadratic Formula.

$$\sin^2 x - 3\sin x - 2 = 0 \quad \text{Write original equation.}$$

$$\sin x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-2)}}{2(1)} \quad \text{Quadratic Formula}$$

$$\sin x = \frac{3 \pm \sqrt{17}}{2} \quad \text{Simplify.}$$

So, $\sin x = \frac{3+\sqrt{17}}{2} \approx 3.5616$ or $\sin x = \frac{3-\sqrt{17}}{2} \approx -0.5616$. The range of the sine function is $[-1, 1]$, so $\sin x = \frac{3+\sqrt{17}}{2}$ has no solution for x. Use a calculator to approximate a solution of $\sin x = \frac{3-\sqrt{17}}{2}$.

$$x = \arcsin\left(\frac{3-\sqrt{17}}{2}\right) \approx -0.5963$$

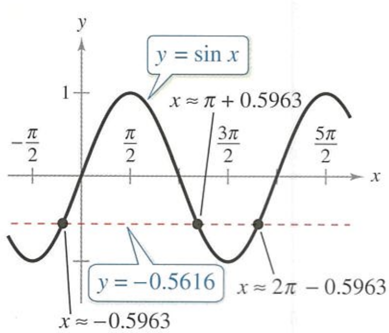

Figure 5.2

Note that this solution is not in the interval $[0, 2\pi)$. To find the solutions in $[0, 2\pi)$, sketch the graphs of $y = \sin x$ and $y = -0.5616$, as shown in Figure 5.2. From the graph, it appears that $\sin x \approx -0.5616$ on the interval $[0, 2\pi)$ when

$$x \approx \pi + 0.5963 \approx 3.7379 \quad \text{and} \quad x \approx 2\pi - 0.5963 \approx 5.6869.$$

So, the solutions of $\sin^2 x - 3\sin x - 2 = 0$ in $[0, 2\pi)$ are $x \approx 3.7379$ and $x \approx 5.6869$.

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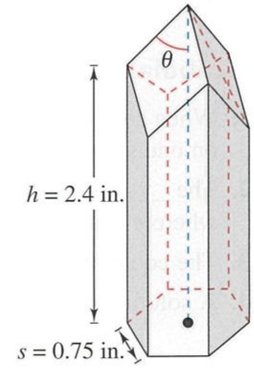
Find all solutions of $\sin^2 x + 2\sin x - 1 = 0$ in the interval $[0, 2\pi)$.

EXAMPLE 11 Surface Area of a Honeycomb Cell

The surface area S (in square inches) of a honeycomb cell is given by

$$S = 6hs + 1.5s^2\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right), \quad 0^\circ < \theta \le 90^\circ$$

where $h = 2.4$ inches, $s = 0.75$ inch, and θ is the angle shown in the figure at the right. What value of θ gives the minimum surface area?

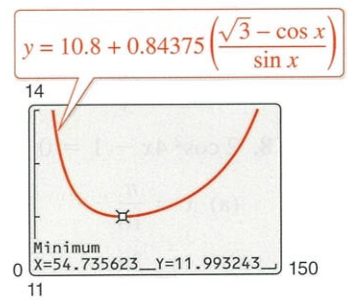


Solution

Letting $h = 2.4$ and $s = 0.75$, you obtain

$$S = 10.8 + 0.84375\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right).$$

Graph this function using a graphing utility set in *degree* mode. Use the *minimum* feature to approximate the minimum point on the graph, as shown in the figure below.



So, the minimum surface area occurs when

$$\theta \approx 54.7356^\circ.$$

REMARK By using calculus, it can be shown that the *exact* minimum surface area occurs when

$$\theta = \arccos\left(\frac{1}{\sqrt{3}}\right).$$

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Use the equation for the surface area of a honeycomb cell given in Example 11 with $h = 3.2$ inches and $s = 0.75$ inch. What value of θ gives the minimum surface area?

Summarize (Section 5.3)

1. Explain how to use standard algebraic techniques to solve trigonometric equations *(page 362)*. For examples of using standard algebraic techniques to solve trigonometric equations, see Examples 1–3.
2. Explain how to solve a trigonometric equation of quadratic type *(page 365)*. For examples of solving trigonometric equations of quadratic type, see Examples 4–6.
3. Explain how to solve a trigonometric equation involving multiple angles *(page 367)*. For examples of solving trigonometric equations involving multiple angles, see Examples 7 and 8.
4. Explain how to use inverse trigonometric functions to solve trigonometric equations *(page 368)*. For examples of using inverse trigonometric functions to solve trigonometric equations, see Examples 9–11.

5.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. When solving a trigonometric equation, the preliminary goal is to ________ the trigonometric function on one side of the equation.
2. The ________ solution of the equation $2 \sin \theta + 1 = 0$ is $\theta = \frac{7\pi}{6} + 2n\pi$ and $\theta = \frac{11\pi}{6} + 2n\pi$, where n is an integer.
3. The equation $2 \tan^2 x - 3 \tan x + 1 = 0$ is a trigonometric equation of ________ type.
4. A solution of an equation that does not satisfy the original equation is an ________ solution.

Skills and Applications

Verifying Solutions In Exercises 5–10, verify that each x-value is a solution of the equation.

5. $\tan x - \sqrt{3} = 0$
 (a) $x = \frac{\pi}{3}$
 (b) $x = \frac{4\pi}{3}$

6. $\sec x - 2 = 0$
 (a) $x = \frac{\pi}{3}$
 (b) $x = \frac{5\pi}{3}$

7. $3 \tan^2 2x - 1 = 0$
 (a) $x = \frac{\pi}{12}$
 (b) $x = \frac{5\pi}{12}$

8. $2 \cos^2 4x - 1 = 0$
 (a) $x = \frac{\pi}{16}$
 (b) $x = \frac{3\pi}{16}$

9. $2 \sin^2 x - \sin x - 1 = 0$
 (a) $x = \frac{\pi}{2}$
 (b) $x = \frac{7\pi}{6}$

10. $\csc^4 x - 4 \csc^2 x = 0$
 (a) $x = \frac{\pi}{6}$
 (b) $x = \frac{5\pi}{6}$

Solving a Trigonometric Equation In Exercises 11–28, solve the equation.

11. $\sqrt{3} \csc x - 2 = 0$
12. $\tan x + \sqrt{3} = 0$
13. $\cos x + 1 = -\cos x$
14. $3 \sin x + 1 = \sin x$
15. $3 \sec^2 x - 4 = 0$
16. $3 \cot^2 x - 1 = 0$
17. $4 \cos^2 x - 1 = 0$
18. $2 - 4 \sin^2 x = 0$
19. $\sin x(\sin x + 1) = 0$
20. $(2 \sin^2 x - 1)(\tan^2 x - 3) = 0$
21. $\cos^3 x - \cos x = 0$
22. $\sec^2 x - 1 = 0$
23. $3 \tan^3 x = \tan x$
24. $\sec x \csc x = 2 \csc x$
25. $2 \cos^2 x + \cos x - 1 = 0$
26. $2 \sin^2 x + 3 \sin x + 1 = 0$
27. $\sec^2 x - \sec x = 2$
28. $\csc^2 x + \csc x = 2$

Solving a Trigonometric Equation In Exercises 29–38, find all solutions of the equation in the interval $[0, 2\pi)$.

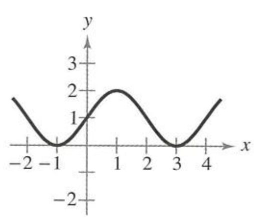
29. $\sin x - 2 = \cos x - 2$
30. $\cos x + \sin x \tan x = 2$
31. $2 \sin^2 x = 2 + \cos x$
32. $\tan^2 x = \sec x - 1$
33. $\sin^2 x = 3 \cos^2 x$
34. $2 \sec^2 x + \tan^2 x - 3 = 0$
35. $2 \sin x + \csc x = 0$
36. $3 \sec x - 4 \cos x = 0$
37. $\csc x + \cot x = 1$
38. $\sec x + \tan x = 1$

Solving a Multiple-Angle Equation In Exercises 39–46, solve the multiple-angle equation.

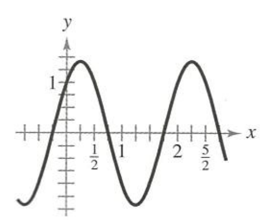
39. $2 \cos 2x - 1 = 0$
40. $2 \sin 2x + \sqrt{3} = 0$
41. $\tan 3x - 1 = 0$
42. $\sec 4x - 2 = 0$
43. $2 \cos \frac{x}{2} - \sqrt{2} = 0$
44. $2 \sin \frac{x}{2} + \sqrt{3} = 0$
45. $3 \tan \frac{x}{2} - \sqrt{3} = 0$
46. $\tan \frac{x}{2} + \sqrt{3} = 0$

Finding x-Intercepts In Exercises 47 and 48, find the x-intercepts of the graph.

47. $y = \sin \frac{\pi x}{2} + 1$



48. $y = \sin \pi x + \cos \pi x$



Approximating Solutions **In Exercises 49–58, use a graphing utility to approximate (to three decimal places) the solutions of the equation in the interval $[0, 2\pi)$.**

49. $5 \sin x + 2 = 0$
50. $2 \tan x + 7 = 0$
51. $\sin x - 3 \cos x = 0$
52. $\sin x + 4 \cos x = 0$
53. $\cos x = x$
54. $\tan x = \csc x$
55. $\sec^2 x - 3 = 0$
56. $\csc^2 x - 5 = 0$
57. $2 \tan^2 x = 15$
58. $6 \sin^2 x = 5$

Using Inverse Functions **In Exercises 59–70, solve the equation.**

59. $\tan^2 x + \tan x - 12 = 0$
60. $\tan^2 x - \tan x - 2 = 0$
61. $\sec^2 x - 6 \tan x = -4$
62. $\sec^2 x + \tan x = 3$
63. $2 \sin^2 x + 5 \cos x = 4$
64. $2 \cos^2 x + 7 \sin x = 5$
65. $\cot^2 x - 9 = 0$
66. $\cot^2 x - 6 \cot x + 5 = 0$
67. $\sec^2 x - 4 \sec x = 0$
68. $\sec^2 x + 2 \sec x - 8 = 0$
69. $\csc^2 x + 3 \csc x - 4 = 0$
70. $\csc^2 x - 5 \csc x = 0$

Using the Quadratic Formula **In Exercises 71–74, use the Quadratic Formula to find all solutions of the equation in the interval $[0, 2\pi)$. Round your result to four decimal places.**

71. $12 \sin^2 x - 13 \sin x + 3 = 0$
72. $3 \tan^2 x + 4 \tan x - 4 = 0$
73. $\tan^2 x + 3 \tan x + 1 = 0$
74. $4 \cos^2 x - 4 \cos x - 1 = 0$

Approximating Solutions **In Exercises 75–78, use a graphing utility to approximate (to three decimal places) the solutions of the equation in the given interval.**

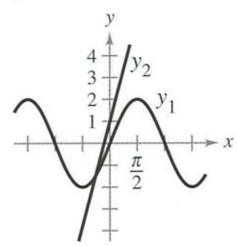
75. $3 \tan^2 x + 5 \tan x - 4 = 0, \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
76. $\cos^2 x - 2 \cos x - 1 = 0, \quad [0, \pi]$
77. $4 \cos^2 x - 2 \sin x + 1 = 0, \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
78. $2 \sec^2 x + \tan x - 6 = 0, \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Approximating Maximum and Minimum Points **In Exercises 79–84, (a) use a graphing utility to graph the function and approximate the maximum and minimum points on the graph in the interval $[0, 2\pi)$, and (b) solve the trigonometric equation and verify that its solutions are the x-coordinates of the maximum and minimum points of f. (Calculus is required to find the trigonometric equation.)**

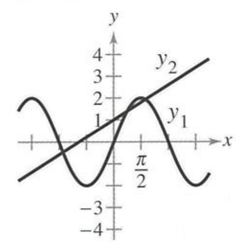
	Function	Trigonometric Equation
79.	$f(x) = \sin^2 x + \cos x$	$2 \sin x \cos x - \sin x = 0$
80.	$f(x) = \cos^2 x - \sin x$	$-2 \sin x \cos x - \cos x = 0$
81.	$f(x) = \sin x + \cos x$	$\cos x - \sin x = 0$
82.	$f(x) = 2 \sin x + \cos 2x$	$2 \cos x - 4 \sin x \cos x = 0$
83.	$f(x) = \sin x \cos x$	$-\sin^2 x + \cos^2 x = 0$
84.	$f(x) = \sec x + \tan x - x$	$\sec x \tan x + \sec^2 x = 1$

Number of Points of Intersection **In Exercises 85 and 86, use the graph to approximate the number of points of intersection of the graphs of y_1 and y_2.**

85. $y_1 = 2 \sin x$
$y_2 = 3x + 1$



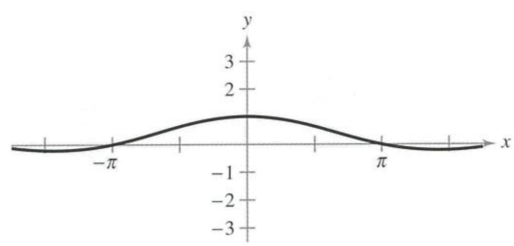
86. $y_1 = 2 \sin x$
$y_2 = \frac{1}{2}x + 1$



87. **Graphical Reasoning** Consider the function

$$f(x) = \frac{\sin x}{x}$$

and its graph, shown in the figure below.



(a) What is the domain of the function?
(b) Identify any symmetry and any asymptotes of the graph.
(c) Describe the behavior of the function as $x \to 0$.
(d) How many solutions does the equation

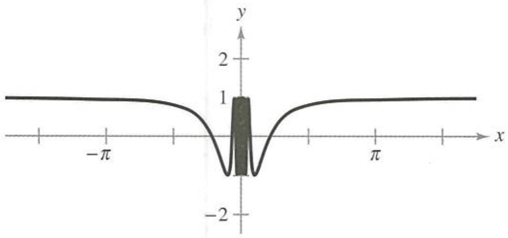
$$\frac{\sin x}{x} = 0$$

have in the interval $[-8, 8]$? Find the solutions.

88. Graphical Reasoning Consider the function

$$f(x) = \cos \frac{1}{x}$$

and its graph, shown in the figure below.



(a) What is the domain of the function?

(b) Identify any symmetry and any asymptotes of the graph.

(c) Describe the behavior of the function as $x \to 0$.

(d) How many solutions does the equation

$$\cos \frac{1}{x} = 0$$

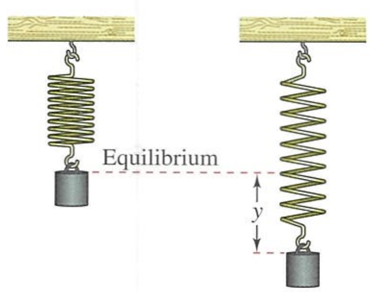
have in the interval $[-1, 1]$? Find the solutions.

(e) Does the equation $\cos(1/x) = 0$ have a greatest solution? If so, then approximate the solution. If not, then explain why.

89. Harmonic Motion A weight is oscillating on the end of a spring (see figure). The displacement from equilibrium of the weight relative to the point of equilibrium is given by

$$y = \tfrac{1}{12}(\cos 8t - 3 \sin 8t)$$

where y is the displacement (in meters) and t is the time (in seconds). Find the times when the weight is at the point of equilibrium ($y = 0$) for $0 \le t \le 1$.



90. Damped Harmonic Motion The displacement from equilibrium of a weight oscillating on the end of a spring is given by

$$y = 1.56e^{-0.22t} \cos 4.9t$$

where y is the displacement (in feet) and t is the time (in seconds). Use a graphing utility to graph the displacement function for $0 \le t \le 10$. Find the time beyond which the distance between the weight and equilibrium does not exceed 1 foot.

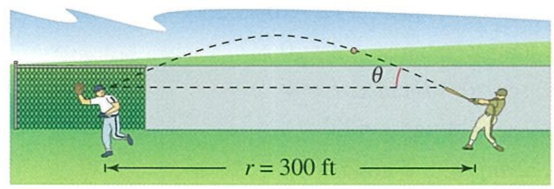
91. Equipment Sales The monthly sales S (in hundreds of units) of skiing equipment at a sports store are approximated by

$$S = 58.3 + 32.5 \cos \frac{\pi t}{6}$$

where t is the time (in months), with $t = 1$ corresponding to January. Determine the months in which sales exceed 7500 units.

92. Projectile Motion A baseball is hit at an angle of θ with the horizontal and with an initial velocity of $v_0 = 100$ feet per second. An outfielder catches the ball 300 feet from home plate (see figure). Find θ when the range r of a projectile is given by

$$r = \frac{1}{32} v_0^2 \sin 2\theta.$$



Not drawn to scale

93. Meteorology The table shows the normal daily high temperatures C in Chicago (in degrees Fahrenheit) for month t, with $t = 1$ corresponding to January. *(Source: NOAA)*

DATA

Month, t	Chicago, C
1	31.0
2	35.3
3	46.6
4	59.0
5	70.0
6	79.7
7	84.1
8	81.9
9	74.8
10	62.3
11	48.2
12	34.8

Spreadsheet at LarsonPrecalculus.com

(a) Use a graphing utility to create a scatter plot of the data.

(b) Find a cosine model for the temperatures.

(c) Graph the model and the scatter plot in the same viewing window. How well does the model fit the data?

(d) What is the overall normal daily high temperature?

(e) Use the graphing utility to determine the months during which the normal daily high temperature is above 72°F and below 72°F.