3.3 Properties of Logarithms

Logarithmic functions have many real-life applications. For example, in Exercises 79–82 on page 224, you will use a logarithmic function that models the relationship between the number of decibels and the intensity of a sound.

- **Use the change-of-base formula to rewrite and evaluate logarithmic expressions.**
- **Use properties of logarithms to evaluate or rewrite logarithmic expressions.**
- **Use properties of logarithms to expand or condense logarithmic expressions.**
- **Use logarithmic functions to model and solve real-life problems.**

Change of Base

Most calculators have only two types of log keys, LOG for common logarithms (base 10) and LN for natural logarithms (base e). Although common logarithms and natural logarithms are the most frequently used, you may occasionally need to evaluate logarithms with other bases. To do this, use the **change-of-base formula.**

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as follows.

Base b	**Base 10**	**Base e**
$\log_a x = \dfrac{\log_b x}{\log_b a}$	$\log_a x = \dfrac{\log x}{\log a}$	$\log_a x = \dfrac{\ln x}{\ln a}$

One way to look at the change-of-base formula is that logarithms with base a are *constant multiples* of logarithms with base b. The constant multiplier is

$$\frac{1}{\log_b a}.$$

EXAMPLE 1 **Changing Bases Using Common Logarithms**

$$\log_4 25 = \frac{\log 25}{\log 4} \qquad \log_a x = \frac{\log x}{\log a}$$

$$\approx \frac{1.39794}{0.60206} \qquad \text{Use a calculator.}$$

$$\approx 2.3219 \qquad \text{Simplify.}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Evaluate $\log_2 12$ using the change-of-base formula and common logarithms.

EXAMPLE 2 **Changing Bases Using Natural Logarithms**

$$\log_4 25 = \frac{\ln 25}{\ln 4} \qquad \log_a x = \frac{\ln x}{\ln a}$$

$$\approx \frac{3.21888}{1.38629} \qquad \text{Use a calculator.}$$

$$\approx 2.3219 \qquad \text{Simplify.}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Evaluate $\log_2 12$ using the change-of-base formula and natural logarithms.

Properties of Logarithms

You know from the preceding section that the logarithmic function with base a is the *inverse function* of the exponential function with base a. So, it makes sense that the properties of exponents have corresponding properties involving logarithms. For example, the exponential property $a^m a^n = a^{m+n}$ has the corresponding logarithmic property $\log_a(uv) = \log_a u + \log_a v$.

REMARK There is no property that can be used to rewrite $\log_a(u \pm v)$. Specifically, $\log_a(u + v)$ is *not* equal to $\log_a u + \log_a v$.

Properties of Logarithms

Let a be a positive number such that $a \neq 1$, let n be a real number, and let u and v be positive real numbers.

	Logarithm with Base a	Natural Logarithm
1. Product Property:	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
2. Quotient Property:	$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$
3. Power Property:	$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$

For proofs of the properties listed above, see Proofs in Mathematics on page 256.

EXAMPLE 3 Using Properties of Logarithms

Write each logarithm in terms of ln 2 and ln 3.

a. $\ln 6$ **b.** $\ln \frac{2}{27}$

Solution

a. $\ln 6 = \ln(2 \cdot 3)$ Rewrite 6 as $2 \cdot 3$.

$= \ln 2 + \ln 3$ Product Property

b. $\ln \frac{2}{27} = \ln 2 - \ln 27$ Quotient Property

$= \ln 2 - \ln 3^3$ Rewrite 27 as 3^3.

$= \ln 2 - 3 \ln 3$ Power Property

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Write each logarithm in terms of log 3 and log 5.

a. $\log 75$ **b.** $\log \frac{9}{125}$

EXAMPLE 4 Using Properties of Logarithms

Find the exact value of $\log_5 \sqrt[3]{5}$ without using a calculator.

Solution

$\log_5 \sqrt[3]{5} = \log_5 5^{1/3} = \frac{1}{3} \log_5 5 = \frac{1}{3}(1) = \frac{1}{3}$

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Find the exact value of $\ln e^6 - \ln e^2$ without using a calculator.

HISTORICAL NOTE

John Napier, a Scottish mathematician, developed logarithms as a way to simplify tedious calculations. Napier worked about 20 years on the development of logarithms before publishing his work is 1614. Napier only partially succeeded in his quest to simplify tedious calculations. Nonetheless, the development of logarithms was a step forward and received immediate recognition.

Rewriting Logarithmic Expressions

The properties of logarithms are useful for rewriting logarithmic expressions in forms that simplify the operations of algebra. This is true because these properties convert complicated products, quotients, and exponential forms into simpler sums, differences, and products, respectively.

EXAMPLE 5 Expanding Logarithmic Expressions

Expand each logarithmic expression.

a. $\log_4 5x^3y$ **b.** $\ln \dfrac{\sqrt{3x-5}}{7}$

Solution

a. $\log_4 5x^3y = \log_4 5 + \log_4 x^3 + \log_4 y$ Product Property

$= \log_4 5 + 3\log_4 x + \log_4 y$ Power Property

b. $\ln \dfrac{\sqrt{3x-5}}{7} = \ln \dfrac{(3x-5)^{1/2}}{7}$ Rewrite using rational exponent.

$= \ln(3x-5)^{1/2} - \ln 7$ Quotient Property

$= \dfrac{1}{2}\ln(3x-5) - \ln 7$ Power Property

▷ **ALGEBRA HELP** To review rewriting radicals and rational exponents, see Appendix A.2.

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Expand the expression $\log_3 \dfrac{4x^2}{\sqrt{y}}$.

Example 5 uses the properties of logarithms to *expand* logarithmic expressions. Example 6 reverses this procedure and uses the properties of logarithms to *condense* logarithmic expressions.

EXAMPLE 6 Condensing Logarithmic Expressions

See LarsonPrecalculus.com for an interactive version of this type of example.

Condense each logarithmic expression.

a. $\frac{1}{2}\log x + 3\log(x+1)$ **b.** $2\ln(x+2) - \ln x$ **c.** $\frac{1}{3}[\log_2 x + \log_2(x+1)]$

Solution

a. $\frac{1}{2}\log x + 3\log(x+1) = \log x^{1/2} + \log(x+1)^3$ Power Property

$= \log\left[\sqrt{x}\,(x+1)^3\right]$ Product Property

b. $2\ln(x+2) - \ln x = \ln(x+2)^2 - \ln x$ Power Property

$= \ln \dfrac{(x+2)^2}{x}$ Quotient Property

c. $\frac{1}{3}[\log_2 x + \log_2(x+1)] = \frac{1}{3}\log_2[x(x+1)]$ Product Property

$= \log_2[x(x+1)]^{1/3}$ Power Property

$= \log_2 \sqrt[3]{x(x+1)}$ Rewrite with a radical.

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Condense the expression $2[\log(x+3) - 2\log(x-2)]$.

Application

One way to determine a possible relationship between the x- and y-values of a set of nonlinear data is to take the natural logarithm of each x-value and each y-value. If the plotted points $(\ln x, \ln y)$ lie on a line, then x and y are related by the equation $\ln y = m \ln x$, where m is the slope of the line.

EXAMPLE 7 Finding a Mathematical Model

The table shows the mean distance x from the sun and the period y (the time it takes a planet to orbit the sun, in years) for each of the six planets that are closest to the sun. In the table, the mean distance is given in astronomical units (where one astronomical unit is defined as Earth's mean distance from the sun). The points from the table are plotted in Figure 3.13. Find an equation that relates y and x.

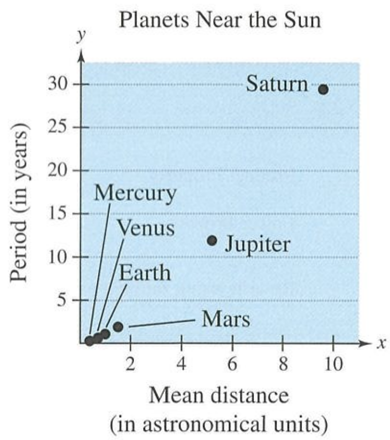

Figure 3.13

DATA

Planet	Mean Distance, x	Period, y
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.524	1.881
Jupiter	5.203	11.862
Saturn	9.537	29.457

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Planet	$\ln x$	$\ln y$
Mercury	−0.949	−1.423
Venus	−0.324	−0.486
Earth	0.000	0.000
Mars	0.421	0.632
Jupiter	1.649	2.473
Saturn	2.255	3.383

Solution From Figure 3.13, it is not clear how to find an equation that relates y and x. To solve this problem, make a table of values giving the natural logarithms of all x- and y-values of the data (see the table at the left). Plot each point $(\ln x, \ln y)$. These points appear to lie on a line (see Figure 3.14). Choose two points to determine the slope of the line. Using the points $(0.421, 0.632)$ and $(0, 0)$, the slope of the line is

$$m = \frac{0.632 - 0}{0.421 - 0} \approx 1.5 = \frac{3}{2}.$$

By the point-slope form, the equation of the line is $Y = \frac{3}{2}X$, where $Y = \ln y$ and $X = \ln x$. So, an equation that relates y and x is $\ln y = \frac{3}{2} \ln x$.

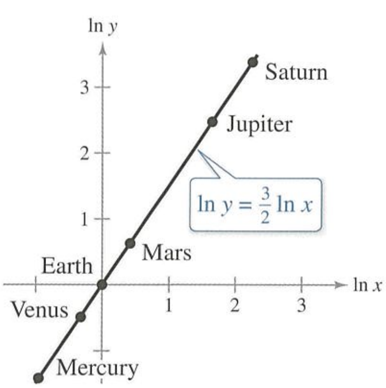

Figure 3.14

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Find a logarithmic equation that relates y and x for the following ordered pairs.

$(0.37, 0.51), (1.00, 1.00), (2.72, 1.95), (7.39, 3.79), (20.09, 7.39)$

Summarize (Section 3.3)

1. State the change-of-base formula *(page 219)*. For examples of using the change-of-base formula to rewrite and evaluate logarithmic expressions, see Examples 1 and 2.
2. Make a list of the properties of logarithms *(page 220)*. For examples of using the properties of logarithms to evaluate or rewrite logarithmic expressions, see Examples 3 and 4.
3. Explain how to use the properties of logarithms to expand or condense logarithmic expressions *(page 221)*. For examples of expanding and condensing logarithmic expressions, see Examples 5 and 6.
4. Describe an example of how to use a logarithmic function to model and solve a real-life problem *(page 222, Example 7)*.

3.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary

In Exercises 1–3, fill in the blanks.

1. To evaluate a logarithm to any base, use the ________ formula.
2. The change-of-base formula for base e is $\log_a x =$ ________.
3. When you consider $\log_a x$ to be a constant multiple of $\log_b x$, the constant multiplier is ________.
4. Name the property of logarithms illustrated by each statement.

(a) $\ln(uv) = \ln u + \ln v$ (b) $\log_a u^n = n \log_a u$ (c) $\ln \frac{u}{v} = \ln u - \ln v$

Skills and Applications

Changing Bases **In Exercises 5–8, rewrite the logarithm as a ratio of (a) common logarithms and (b) natural logarithms.**

5. $\log_5 16$
6. $\log_{1/5} 4$
7. $\log_x \frac{3}{10}$
8. $\log_{2.6} x$

Using the Change-of-Base Formula **In Exercises 9–12, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.**

9. $\log_3 17$
10. $\log_{0.4} 12$
11. $\log_\pi 0.5$
12. $\log_{2/3} 0.125$

Using Properties of Logarithms **In Exercises 13–18, use the properties of logarithms to write the logarithm in terms of $\log_3 5$ and $\log_3 7$.**

13. $\log_3 35$
14. $\log_3 \frac{5}{7}$
15. $\log_3 \frac{7}{25}$
16. $\log_3 175$
17. $\log_3 \frac{21}{5}$
18. $\log_3 \frac{45}{49}$

Using Properties of Logarithms **In Exercises 19–32, find the exact value of the logarithmic expression without using a calculator. (If this is not possible, state the reason.)**

19. $\log_3 9$
20. $\log_5 \frac{1}{125}$
21. $\log_6 \sqrt[3]{\frac{1}{6}}$
22. $\log_2 \sqrt[4]{8}$
23. $\log_2(-2)$
24. $\log_3(-27)$
25. $\ln \sqrt[4]{e^3}$
26. $\ln(1/\sqrt{e})$
27. $\ln e^2 + \ln e^5$
28. $2 \ln e^6 - \ln e^5$
29. $\log_5 75 - \log_5 3$
30. $\log_4 2 + \log_4 32$
31. $\log_4 8$
32. $\log_8 16$

Using Properties of Logarithms **In Exercises 33–40, approximate the logarithm using the properties of logarithms, given $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$.**

33. $\log_b 10$
34. $\log_b \frac{2}{3}$
35. $\log_b 0.04$
36. $\log_b \sqrt{2}$
37. $\log_b 45$
38. $\log_b(3b^2)$
39. $\log_b(2b)^{-2}$
40. $\log_b \sqrt[3]{3b}$

Expanding a Logarithmic Expression **In Exercises 41–60, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)**

41. $\ln 7x$
42. $\log_3 13z$
43. $\log_8 x^4$
44. $\ln(xy)^3$
45. $\log_5 \frac{5}{x}$
46. $\log_6 \frac{w^2}{v}$
47. $\ln \sqrt{z}$
48. $\ln \sqrt[3]{t}$
49. $\ln xyz^2$
50. $\log_4 11b^2c$
51. $\ln z(z-1)^2, \quad z > 1$
52. $\ln \frac{x^2 - 1}{x^3}, \quad x > 1$
53. $\log_2 \frac{\sqrt{a^2 - 4}}{7}, \quad a > 2$
54. $\ln \frac{3}{\sqrt{x^2 + 1}}$
55. $\log_5 \frac{x^2}{y^2 z^3}$
56. $\log_{10} \frac{xy^4}{z^5}$
57. $\ln \sqrt[3]{\frac{yz}{x^2}}$
58. $\log_2 x^4 \sqrt{\frac{y}{z^3}}$
59. $\ln \sqrt[4]{x^3(x^2 + 3)}$
60. $\ln \sqrt{x^2(x + 2)}$

Condensing a Logarithmic Expression **In Exercises 61–76, condense the expression to the logarithm of a single quantity.**

61. $\ln 3 + \ln x$

62. $\log_5 8 - \log_5 t$

63. $\frac{2}{3} \log_7(z - 2)$

64. $-4 \ln 3x$

65. $\log_3 5x - 4 \log_3 x$

66. $2 \log_2 x + 4 \log_2 y$

67. $\log x + 2 \log(x + 1)$

68. $2 \ln 8 - 5 \ln(z - 4)$

69. $\log x - 2 \log y + 3 \log z$

70. $3 \log_3 x + \frac{1}{4} \log_3 y - 4 \log_3 z$

71. $\ln x - [\ln(x + 1) + \ln(x - 1)]$

72. $4[\ln z + \ln(z + 5)] - 2 \ln(z - 5)$

73. $\frac{1}{2}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)]$

74. $2[3 \ln x - \ln(x + 1) - \ln(x - 1)]$

75. $\frac{1}{3}[\log_8 y + 2 \log_8(y + 4)] - \log_8(y - 1)$

76. $\frac{1}{2}[\log_4(x + 1) + 2 \log_4(x - 1)] + 6 \log_4 x$

Comparing Logarithmic Quantities **In Exercises 77 and 78, determine which (if any) of the logarithmic expressions are equal. Justify your answer.**

77. $\dfrac{\log_2 32}{\log_2 4}$, $\log_2 \dfrac{32}{4}$, $\log_2 32 - \log_2 4$

78. $\log_7 \sqrt{70}$, $\log_7 35$, $\frac{1}{2} + \log_7 \sqrt{10}$

Sound Intensity

In Exercises 79–82, use the following information. The relationship between the number of decibels β and the intensity of a sound I (in watts per square meter) is

$$\beta = 10 \log \frac{I}{10^{-12}}.$$

79. Use the properties of logarithms to write the formula in a simpler form. Then determine the number of decibels of a sound with an intensity of 10^{-6} watt per square meter.

80. Find the difference in loudness between an average office with an intensity of 1.26×10^{-7} watt per square meter and a broadcast studio with an intensity of 3.16×10^{-10} watt per square meter.

81. Find the difference in loudness between a vacuum cleaner with an intensity of 10^{-4} watt per square meter and rustling leaves with an intensity of 10^{-11} watt per square meter.

82. You and your roommate are playing your stereos at the same time and at the same intensity. How much louder is the music when both stereos are playing compared with just one stereo playing?

Curve Fitting **In Exercises 83–86, find a logarithmic equation that relates y and x.**

83.

x	1	2	3	4	5	6
y	1	1.189	1.316	1.414	1.495	1.565

84.

x	1	2	3	4	5	6
y	1	0.630	0.481	0.397	0.342	0.303

85.

x	1	2	3	4	5	6
y	2.5	2.102	1.9	1.768	1.672	1.597

86.

x	1	2	3	4	5	6
y	0.5	2.828	7.794	16	27.951	44.091

87. Stride Frequency of Animals Four-legged animals run with two different types of motion: trotting and galloping. An animal that is trotting has at least one foot on the ground at all times, whereas an animal that is galloping has all four feet off the ground at some point in its stride. The number of strides per minute at which an animal breaks from a trot to a gallop depends on the weight of the animal. Use the table to find a logarithmic equation that relates an animal's weight x (in pounds) and its lowest stride frequency while galloping y (in strides per minute).

DATA

Weight, x	Stride Frequency, y
25	191.5
35	182.7
50	173.8
75	164.2
500	125.9
1000	114.2

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88. Nail Length The approximate lengths and diameters (in inches) of bright common wire nails are shown in the table. Find a logarithmic equation that relates the diameter y of a bright common wire nail to its length x.

Length, x	Diameter, y
2	0.113
3	0.148
4	0.192
5	0.225
6	0.262

89. Comparing Models A cup of water at an initial temperature of 78°C is placed in a room at a constant temperature of 21°C. The temperature of the water is measured every 5 minutes during a half-hour period. The results are recorded as ordered pairs of the form (t, T), where t is the time (in minutes) and T is the temperature (in degrees Celsius).

$(0, 78.0°)$, $(5, 66.0°)$, $(10, 57.5°)$, $(15, 51.2°)$, $(20, 46.3°)$, $(25, 42.4°)$, $(30, 39.6°)$

(a) Subtract the room temperature from each of the temperatures in the ordered pairs. Use a graphing utility to plot the data points (t, T) and $(t, T - 21)$.

(b) An exponential model for the data $(t, T - 21)$ is $T - 21 = 54.4(0.964)^t$. Solve for T and graph the model. Compare the result with the plot of the original data.

(c) Use the graphing utility to plot the points $(t, \ln(T - 21))$ and observe that the points appear to be linear. Use the *regression* feature of the graphing utility to fit a line to these data. This resulting line has the form $\ln(T - 21) = at + b$, which is equivalent to $e^{\ln(T-21)} = e^{at+b}$. Solve for T, and verify that the result is equivalent to the model in part (b).

(d) Fit a rational model to the data. Take the reciprocals of the y-coordinates of the revised data points to generate the points

$$\left(t, \frac{1}{T - 21}\right).$$

Use the graphing utility to graph these points and observe that they appear to be linear. Use the *regression* feature of the graphing utility to fit a line to these data. The resulting line has the form

$$\frac{1}{T - 21} = at + b.$$

Solve for T, and use the graphing utility to graph the rational function and the original data points.

90. Writing Write a short paragraph explaining why the transformations of the data in Exercise 89 were necessary to obtain the models. Why did taking the logarithms of the temperatures lead to a linear scatter plot? Why did taking the reciprocals of the temperatures lead to a linear scatter plot?

Exploration

True or False? **In Exercises 91–96, determine whether the statement is true or false given that $f(x) = \ln x$. Justify your answer.**

91. $f(0) = 0$

92. $f(ax) = f(a) + f(x), \quad a > 0, \quad x > 0$

93. $f(x - 2) = f(x) - f(2), \quad x > 2$

94. $\sqrt{f(x)} = \frac{1}{2}f(x)$

95. If $f(u) = 2f(v)$, then $v = u^2$.

96. If $f(x) < 0$, then $0 < x < 1$.

Using the Change-of-Base Formula **In Exercises 97–100, use the change-of-base formula to rewrite the logarithm as a ratio of logarithms. Then use a graphing utility to graph the ratio.**

97. $f(x) = \log_2 x$

98. $f(x) = \log_{1/2} x$

99. $f(x) = \log_{1/4} x$

100. $f(x) = \log_{11.8} x$

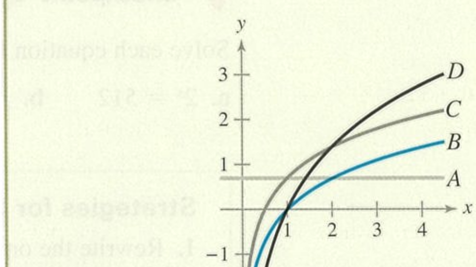
Error Analysis **In Exercises 101 and 102, describe the error.**

101. $(\ln e)^2 = 2(\ln e) = 2(1) = 2$ ✗

102. $\log_2 8 = \log_2(4 + 4)$
$= \log_2 4 + \log_2 4$
$= \log_2 2^2 + \log_2 2^2$
$= 2 + 2$
$= 4$ ✗

103. Graphical Reasoning Use a graphing utility to graph the functions $y_1 = \ln x - \ln(x - 3)$ and $y_2 = \ln \dfrac{x}{x - 3}$ in the same viewing window. Does the graphing utility show the functions with the same domain? If not, explain why some numbers are in the domain of one function but not the other.

104. HOW DO YOU SEE IT? The figure shows the graphs of $y = \ln x$, $y = \ln x^2$, $y = \ln 2x$, and $y = \ln 2$. Match each function with its graph. (The graphs are labeled A through D.) Explain.



105. Think About It For which integers between 1 and 20 can you approximate natural logarithms, given the values $\ln 2 \approx 0.6931$, $\ln 3 \approx 1.0986$, and $\ln 5 \approx 1.6094$? Approximate these logarithms. (Do not use a calculator.)

3.4 Exponential and Logarithmic Equations

Exponential and logarithmic equations have many life science applications. For example, Exercise 83 on page 234 uses an exponential function to model the beaver population in a given area.

- **Solve simple exponential and logarithmic equations.**
- **Solve more complicated exponential equations.**
- **Solve more complicated logarithmic equations.**
- **Use exponential and logarithmic equations to model and solve real-life problems.**

Introduction

So far in this chapter, you have studied the definitions, graphs, and properties of exponential and logarithmic functions. In this section, you will study procedures for *solving equations* involving exponential and logarithmic expressions.

There are two basic strategies for solving exponential or logarithmic equations. The first is based on the One-to-One Properties and was used to solve simple exponential and logarithmic equations in Sections 3.1 and 3.2. The second is based on the Inverse Properties. For $a > 0$ and $a \neq 1$, the properties below are true for all x and y for which $\log_a x$ and $\log_a y$ are defined.

One-to-One Properties	**Inverse Properties**
$a^x = a^y$ if and only if $x = y$.	$a^{\log_a x} = x$
$\log_a x = \log_a y$ if and only if $x = y$.	$\log_a a^x = x$

EXAMPLE 1 **Solving Simple Equations**

	Original Equation	Rewritten Equation	Solution	Property
a.	$2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
b.	$\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
c.	$\left(\frac{1}{3}\right)^x = 9$	$3^{-x} = 3^2$	$x = -2$	One-to-One
d.	$e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
e.	$\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
f.	$\log x = -1$	$10^{\log x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse
g.	$\log_3 x = 4$	$3^{\log_3 x} = 3^4$	$x = 81$	Inverse

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Solve each equation for x.

a. $2^x = 512$ **b.** $\log_6 x = 3$ **c.** $5 - e^x = 0$ **d.** $9^x = \frac{1}{3}$

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

Solving Exponential Equations

EXAMPLE 2 **Solving Exponential Equations**

Solve each equation and approximate the result to three decimal places, if necessary.

a. $e^{-x^2} = e^{-3x-4}$ **b.** $3(2^x) = 42$

Solution

a.
$$e^{-x^2} = e^{-3x-4} \quad \text{Write original equation.}$$
$$-x^2 = -3x - 4 \quad \text{One-to-One Property}$$
$$x^2 - 3x - 4 = 0 \quad \text{Write in general form.}$$
$$(x+1)(x-4) = 0 \quad \text{Factor.}$$
$$x + 1 = 0 \Rightarrow x = -1 \quad \text{Set 1st factor equal to 0.}$$
$$x - 4 = 0 \Rightarrow x = 4 \quad \text{Set 2nd factor equal to 0.}$$

The solutions are $x = -1$ and $x = 4$. Check these in the original equation.

b.
$$3(2^x) = 42 \quad \text{Write original equation.}$$
$$2^x = 14 \quad \text{Divide each side by 3.}$$
$$\log_2 2^x = \log_2 14 \quad \text{Take log (base 2) of each side.}$$
$$x = \log_2 14 \quad \text{Inverse Property}$$
$$x = \frac{\ln 14}{\ln 2} \approx 3.807 \quad \text{Change-of-base formula}$$

The solution is $x = \log_2 14 \approx 3.807$. Check this in the original equation.

REMARK Another way to solve Example 2(b) is by taking the natural log of each side and then applying the Power Property.

$$3(2^x) = 42$$
$$2^x = 14$$
$$\ln 2^x = \ln 14$$
$$x \ln 2 = \ln 14$$
$$x = \frac{\ln 14}{\ln 2} \approx 3.807$$

Notice that you obtain the same result as in Example 2(b).

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Solve each equation and approximate the result to three decimal places, if necessary.

a. $e^{2x} = e^{x^2-8}$ **b.** $2(5^x) = 32$

In Example 2(b), the exact solution is $x = \log_2 14$, and the approximate solution is $x \approx 3.807$. An exact answer is preferred when the solution is an intermediate step in a larger problem. For a final answer, an approximate solution is more practical.

EXAMPLE 3 **Solving an Exponential Equation**

Solve $e^x + 5 = 60$ and approximate the result to three decimal places.

Solution

$$e^x + 5 = 60 \quad \text{Write original equation.}$$
$$e^x = 55 \quad \text{Subtract 5 from each side.}$$
$$\ln e^x = \ln 55 \quad \text{Take natural log of each side.}$$
$$x = \ln 55 \approx 4.007 \quad \text{Inverse Property}$$

The solution is $x = \ln 55 \approx 4.007$. Check this in the original equation.

REMARK Remember that the natural logarithmic function has a base of e.

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Solve $e^x - 7 = 23$ and approximate the result to three decimal places.

EXAMPLE 4 **Solving an Exponential Equation**

Solve $2(3^{2t-5}) - 4 = 11$ and approximate the result to three decimal places.

Solution

$2(3^{2t-5}) - 4 = 11$ — Write original equation.

$2(3^{2t-5}) = 15$ — Add 4 to each side.

$3^{2t-5} = \frac{15}{2}$ — Divide each side by 2.

$\log_3 3^{2t-5} = \log_3 \frac{15}{2}$ — Take log (base 3) of each side.

$2t - 5 = \log_3 \frac{15}{2}$ — Inverse Property

$2t = 5 + \log_3 7.5$ — Add 5 to each side.

$t = \frac{5}{2} + \frac{1}{2}\log_3 7.5$ — Divide each side by 2.

$t \approx 3.417$ — Use a calculator.

The solution is $t = \frac{5}{2} + \frac{1}{2}\log_3 7.5 \approx 3.417$. Check this in the original equation.

REMARK Remember that to evaluate a logarithm such as $\log_3 7.5$, you need to use the change-of-base formula.

$$\log_3 7.5 = \frac{\ln 7.5}{\ln 3} \approx 1.834$$

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Solve $6(2^{t+5}) + 4 = 11$ and approximate the result to three decimal places.

When an equation involves two or more exponential expressions, you can still use a procedure similar to that demonstrated in Examples 2, 3, and 4. However, it may include additional algebraic techniques.

EXAMPLE 5 **Solving an Exponential Equation of Quadratic Type**

Solve $e^{2x} - 3e^x + 2 = 0$.

Algebraic Solution

$e^{2x} - 3e^x + 2 = 0$ — Write original equation.

$(e^x)^2 - 3e^x + 2 = 0$ — Write in quadratic form.

$(e^x - 2)(e^x - 1) = 0$ — Factor.

$e^x - 2 = 0$ — Set 1st factor equal to 0.

$x = \ln 2$ — Solve for x.

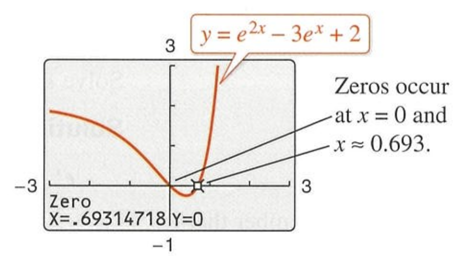
$e^x - 1 = 0$ — Set 2nd factor equal to 0.

$x = 0$ — Solve for x.

The solutions are $x = \ln 2 \approx 0.693$ and $x = 0$. Check these in the original equation.

Graphical Solution

Use a graphing utility to graph $y = e^{2x} - 3e^x + 2$ and then find the zeros.



So, the solutions are $x = 0$ and $x \approx 0.693$.

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Solve $e^{2x} - 7e^x + 12 = 0$.

Solving Logarithmic Equations

To solve a logarithmic equation, write it in exponential form. This procedure is called *exponentiating* each side of an equation.

$\ln x = 3$ Logarithmic form

$e^{\ln x} = e^3$ Exponentiate each side.

$x = e^3$ Exponential form

REMARK When solving equations, remember to check your solutions in the original equation to verify that the answer is correct and to make sure that the answer is in the domain of the original equation.

EXAMPLE 6 Solving Logarithmic Equations

a. $\ln x = 2$ Original equation

$e^{\ln x} = e^2$ Exponentiate each side.

$x = e^2$ Inverse Property

b. $\log_3(5x - 1) = \log_3(x + 7)$ Original equation

$5x - 1 = x + 7$ One-to-One Property

$x = 2$ Solve for x.

c. $\log_6(3x + 14) - \log_6 5 = \log_6 2x$ Original equation

$\log_6\left(\frac{3x + 14}{5}\right) = \log_6 2x$ Quotient Property of Logarithms

$\frac{3x + 14}{5} = 2x$ One-to-One Property

$3x + 14 = 10x$ Multiply each side by 5.

$x = 2$ Solve for x.

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Solve each equation.

a. $\ln x = \frac{2}{3}$ **b.** $\log_2(2x - 3) = \log_2(x + 4)$ **c.** $\log 4x - \log(12 + x) = \log 2$

EXAMPLE 7 Solving a Logarithmic Equation

Solve $5 + 2 \ln x = 4$ and approximate the result to three decimal places.

Algebraic Solution

$5 + 2 \ln x = 4$ Write original equation.

$2 \ln x = -1$ Subtract 5 from each side.

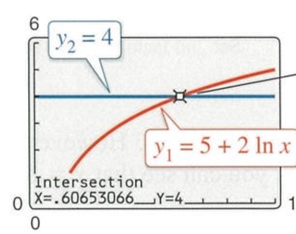
$\ln x = -\frac{1}{2}$ Divide each side by 2.

$e^{\ln x} = e^{-1/2}$ Exponentiate each side.

$x = e^{-1/2}$ Inverse Property

$x \approx 0.607$ Use a calculator.

Graphical Solution



The intersection point is about (0.607, 4).

So, the solution is $x \approx 0.607$.

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Solve $7 + 3 \ln x = 5$ and approximate the result to three decimal places.

EXAMPLE 8 Solving a Logarithmic Equation

Solve $2 \log_5 3x = 4$.

Solution

$2 \log_5 3x = 4$	Write original equation.
$\log_5 3x = 2$	Divide each side by 2.
$5^{\log_5 3x} = 5^2$	Exponentiate each side (base 5).
$3x = 25$	Inverse Property
$x = \frac{25}{3}$	Divide each side by 3.

The solution is $x = \frac{25}{3}$. Check this in the original equation.

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Solve $3 \log_4 6x = 9$.

The domain of a logarithmic function generally does not include all real numbers, so you should be sure to check for extraneous solutions of logarithmic equations.

EXAMPLE 9 Checking for Extraneous Solutions

Solve

$$\log 5x + \log(x - 1) = 2.$$

Algebraic Solution

$\log 5x + \log(x - 1) = 2$	Write original equation.
$\log[5x(x - 1)] = 2$	Product Property of Logarithms
$10^{\log(5x^2 - 5x)} = 10^2$	Exponentiate each side (base 10).
$5x^2 - 5x = 100$	Inverse Property
$x^2 - x - 20 = 0$	Write in general form.
$(x - 5)(x + 4) = 0$	Factor.
$x - 5 = 0$	Set 1st factor equal to 0.
$x = 5$	Solve for x.
$x + 4 = 0$	Set 2nd factor equal to 0.
$x = -4$	Solve for x.

The solutions appear to be $x = 5$ and $x = -4$. However, when you check these in the original equation, you can see that $x = 5$ is the only solution.

Graphical Solution

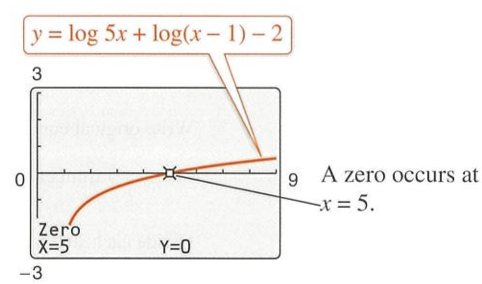
First, rewrite the original equation as

$$\log 5x + \log(x - 1) - 2 = 0.$$

Then use a graphing utility to graph the equation

$$y = \log 5x + \log(x - 1) - 2$$

and find the zero(s).



So, the solution is $x = 5$.

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Solve $\log x + \log(x - 9) = 1$.

In Example 9, the domain of $\log 5x$ is $x > 0$ and the domain of $\log(x - 1)$ is $x > 1$, so the domain of the original equation is $x > 1$. This means that the solution $x = -4$ is extraneous. The graphical solution verifies this conclusion.

Applications

EXAMPLE 10 Doubling an Investment

See LarsonPrecalculus.com for an interactive version of this type of example.

You invest \$500 at an annual interest rate of 6.75%, compounded continuously. How long will it take your money to double?

Solution Using the formula for continuous compounding, the balance is

$$A = Pe^{rt}$$

$$A = 500e^{0.0675t}.$$

To find the time required for the balance to double, let $A = 1000$ and solve the resulting equation for t.

$$500e^{0.0675t} = 1000 \quad \text{Let } A = 1000.$$

$$e^{0.0675t} = 2 \quad \text{Divide each side by 500.}$$

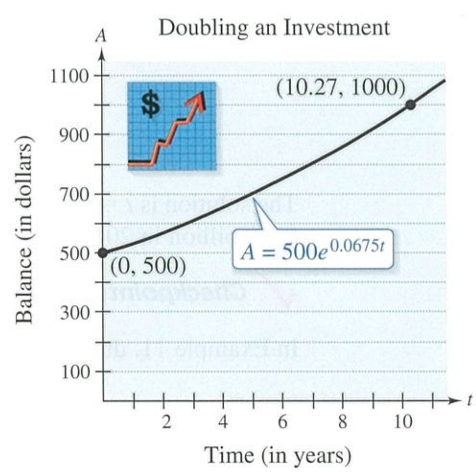
$$\ln e^{0.0675t} = \ln 2 \quad \text{Take natural log of each side.}$$

$$0.0675t = \ln 2 \quad \text{Inverse Property}$$

$$t = \frac{\ln 2}{0.0675} \quad \text{Divide each side by 0.0675.}$$

$$t \approx 10.27 \quad \text{Use a calculator.}$$

The balance in the account will double after approximately 10.27 years. This result is demonstrated graphically below.



✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

You invest \$500 at an annual interest rate of 5.25%, compounded continuously. How long will it take your money to double? Compare your result with that of Example 10.

In Example 10, an approximate answer of 10.27 years is given. Within the context of the problem, the exact solution

$$t = \frac{\ln 2}{0.0675}$$

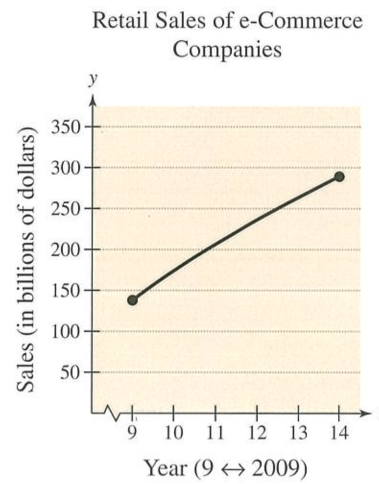
does not make sense as an answer.

EXAMPLE 11 Retail Sales

The retail sales y (in billions of dollars) of e-commerce companies in the United States from 2009 through 2014 can be modeled by

$$y = -614 + 342.2 \ln t, \quad 9 \le t \le 14$$

where t represents the year, with $t = 9$ corresponding to 2009 (see figure). During which year did the sales reach \$240 billion? *(Source: U.S. Census Bureau)*



Solution

$$-614 + 342.2 \ln t = y \qquad \text{Write original equation.}$$

$$-614 + 342.2 \ln t = 240 \qquad \text{Substitute 240 for } y.$$

$$342.2 \ln t = 854 \qquad \text{Add 614 to each side.}$$

$$\ln t = \frac{854}{342.2} \qquad \text{Divide each side by 342.2.}$$

$$e^{\ln t} = e^{854/342.2} \qquad \text{Exponentiate each side.}$$

$$t = e^{854/342.2} \qquad \text{Inverse Property}$$

$$t \approx 12 \qquad \text{Use a calculator.}$$

The solution is $t \approx 12$. Because $t = 9$ represents 2009, it follows that the sales reached \$240 billion in 2012.

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In Example 11, during which year did the sales reach \$180 billion?

Summarize (Section 3.4)

1. State the One-to-One Properties and the Inverse Properties that are used to solve simple exponential and logarithmic equations *(page 226)*. For an example of solving simple exponential and logarithmic equations, see Example 1.
2. Describe strategies for solving exponential equations *(pages 227 and 228)*. For examples of solving exponential equations, see Examples 2–5.
3. Describe strategies for solving logarithmic equations *(pages 229 and 230)*. For examples of solving logarithmic equations, see Examples 6–9.
4. Describe examples of how to use exponential and logarithmic equations to model and solve real-life problems *(pages 231 and 232, Examples 10 and 11)*.

3.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. To solve exponential and logarithmic equations, you can use the One-to-One and Inverse Properties below.

(a) $a^x = a^y$ if and only if ________.
(b) $\log_a x = \log_a y$ if and only if ________.
(c) $a^{\log_a x} =$ ________
(d) $\log_a a^x =$ ________

2. An ________ solution does not satisfy the original equation.

Skills and Applications

Determining Solutions **In Exercises 3–6, determine whether each x-value is a solution (or an approximate solution) of the equation.**

3. $4^{2x-7} = 64$
(a) $x = 5$
(b) $x = 2$
(c) $x = \frac{1}{2}(\log_4 64 + 7)$

4. $4e^{x-1} = 60$
(a) $x = 1 + \ln 15$
(b) $x \approx 1.708$
(c) $x = \ln 16$

5. $\log_2(x + 3) = 10$
(a) $x = 1021$
(b) $x = 17$
(c) $x = 10^2 - 3$

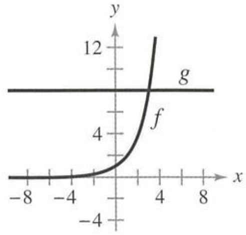
6. $\ln(2x + 3) = 5.8$
(a) $x = \frac{1}{2}(-3 + \ln 5.8)$
(b) $x = \frac{1}{2}(-3 + e^{5.8})$
(c) $x \approx 163.650$

Solving a Simple Equation **In Exercises 7–16, solve for x.**

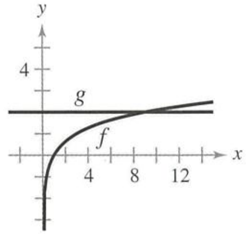
7. $4^x = 16$
8. $\left(\frac{1}{2}\right)^x = 32$
9. $\ln x - \ln 2 = 0$
10. $\log x - \log 10 = 0$
11. $e^x = 2$
12. $e^x = \frac{1}{3}$
13. $\ln x = -1$
14. $\log x = -2$
15. $\log_4 x = 3$
16. $\log_5 x = \frac{1}{2}$

Approximating a Point of Intersection **In Exercises 17 and 18, approximate the point of intersection of the graphs of f and g. Then solve the equation $f(x) = g(x)$ algebraically to verify your approximation.**

17. $f(x) = 2^x,\ g(x) = 8$



18. $f(x) = \log_3 x,\ g(x) = 2$



Solving an Exponential Equation **In Exercises 19–46, solve the exponential equation algebraically. Approximate the result to three decimal places, if necessary.**

19. $e^x = e^{x^2-2}$
20. $e^{x^2-3} = e^{x-2}$
21. $4(3^x) = 20$
22. $4e^x = 91$
23. $e^x - 8 = 31$
24. $5^x + 8 = 26$
25. $3^{2x} = 80$
26. $4^{-3t} = 0.10$
27. $3^{2-x} = 400$
28. $7^{-3-x} = 242$
29. $8(10^{3x}) = 12$
30. $8(3^{6-x}) = 40$
31. $e^{3x} = 12$
32. $500e^{-2x} = 125$
33. $7 - 2e^x = 5$
34. $-14 + 3e^x = 11$
35. $6(2^{3x-1}) - 7 = 9$
36. $8(4^{6-2x}) + 13 = 41$
37. $3^x = 2^{x-1}$
38. $e^{x+1} = 2^{x+2}$
39. $4^x = 5^{x^2}$
40. $3^{x^2} = 7^{6-x}$
41. $e^{2x} - 4e^x - 5 = 0$
42. $e^{2x} - 5e^x + 6 = 0$
43. $\dfrac{1}{1 - e^x} = 5$
44. $\dfrac{100}{1 + e^{2x}} = 1$
45. $\left(1 + \dfrac{0.065}{365}\right)^{365t} = 4$
46. $\left(1 + \dfrac{0.10}{12}\right)^{12t} = 2$

Solving a Logarithmic Equation **In Exercises 47–62, solve the logarithmic equation algebraically. Approximate the result to three decimal places, if necessary.**

47. $\ln x = -3$
48. $\ln x - 7 = 0$
49. $2.1 = \ln 6x$
50. $\log 3z = 2$
51. $3 - 4 \ln x = 11$
52. $3 + 8 \ln x = 7$
53. $6 \log_3 0.5x = 11$
54. $4 \log(x - 6) = 11$
55. $\ln x - \ln(x + 1) = 2$
56. $\ln x + \ln(x + 1) = 1$
57. $\ln(x + 5) = \ln(x - 1) - \ln(x + 1)$
58. $\ln(x + 1) - \ln(x - 2) = \ln x$
59. $\log(3x + 4) = \log(x - 10)$
60. $\log_2 x + \log_2(x + 2) = \log_2(x + 6)$
61. $\log_4 x - \log_4(x - 1) = \frac{1}{2}$
62. $\log 8x - \log(1 + \sqrt{x}) = 2$

Using Technology **In Exercises 63–70, use a graphing utility to graphically solve the equation. Approximate the result to three decimal places. Verify your result algebraically.**

63. $5^x = 212$

64. $6e^{1-x} = 25$

65. $8e^{-2x/3} = 11$

66. $e^{0.09t} = 3$

67. $3 - \ln x = 0$

68. $10 - 4\ln(x - 2) = 0$

69. $2\ln(x + 3) = 3$

70. $\ln(x + 1) = 2 - \ln x$

Compound Interest **In Exercises 71 and 72, you invest \$2500 in an account at interest rate *r*, compounded continuously. Find the time required for the amount to (a) double and (b) triple.**

71. $r = 0.025$

72. $r = 0.0375$

Algebra of Calculus **In Exercises 73–80, solve the equation algebraically. Round your result to three decimal places, if necessary. Verify your answer using a graphing utility.**

73. $2x^2e^{2x} + 2xe^{2x} = 0$

74. $-x^2e^{-x} + 2xe^{-x} = 0$

75. $-xe^{-x} + e^{-x} = 0$

76. $e^{-2x} - 2xe^{-2x} = 0$

77. $\dfrac{1 + \ln x}{2} = 0$

78. $\dfrac{1 - \ln x}{x^2} = 0$

79. $2x\ln x + x = 0$

80. $2x\ln\left(\dfrac{1}{x}\right) - x = 0$

81. Average Heights The percent m of American males between the ages of 20 and 29 who are under x inches tall is modeled by

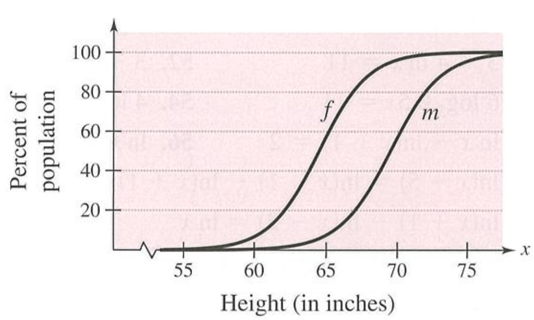
$$m(x) = \frac{100}{1 + e^{-0.5536(x-69.51)}}, \quad 64 \le x \le 78$$

and the percent f of American females between the ages of 20 and 29 who are under x inches tall is modeled by

$$f(x) = \frac{100}{1 + e^{-0.5834(x-64.49)}}, \quad 60 \le x \le 78.$$

(Source: U.S. National Center for Health Statistics)

(a) Use the graph to determine any horizontal asymptotes of the graphs of the functions. Interpret the meaning in the context of the problem.



(b) What is the average height of each sex?

82. Demand The demand equation for a smartphone is

$$p = 5000\left(1 - \frac{4}{4 + e^{-0.002x}}\right).$$

Find the demand x for each price.

(a) $p = \$169$

(b) $p = \$299$

83. Ecology The number N of beavers in a given area after x years can be approximated by

$$N = 5.5 \cdot 10^{0.23x}, \quad 0 \le x \le 10.$$

Use the model to approximate how many years it will take for the beaver population to reach 78.

84. Ecology The number N of trees of a given species per acre is approximated by the model

$$N = 3500(10^{-0.12x}), \quad 3 \le x \le 30$$

where x is the average diameter of the trees (in inches) 4.5 feet above the ground. Use the model to approximate the average diameter of the trees in a test plot when $N = 22$.

85. Population The population P (in thousands) of Alaska in the years 2005 through 2015 can be modeled by

$$P = 75\ln t + 540, \quad 5 \le t \le 15$$

where t represents the year, with $t = 5$ corresponding to 2005. During which year did the population of Alaska exceed 720 thousand? *(Source: U.S. Census Bureau)*

86. Population The population P (in thousands) of Montana in the years 2005 through 2015 can be modeled by

$$P = 81\ln t + 807, \quad 5 \le t \le 15$$

where t represents the year, with $t = 5$ corresponding to 2005. During which year did the population of Montana exceed 965 thousand? *(Source: U.S. Census Bureau)*

87. Temperature An object at a temperature of 80°C is placed in a room at 20°C. The temperature of the object is given by

$$T = 20 + 60e^{-0.06m}$$

where m represents the number of minutes after the object is placed in the room. How long does it take the object to reach a temperature of 70°C?

88. Temperature An object at a temperature of 160°C was removed from a furnace and placed in a room at 20°C. The temperature T of the object was measured each hour h and recorded in the table. A model for the data is

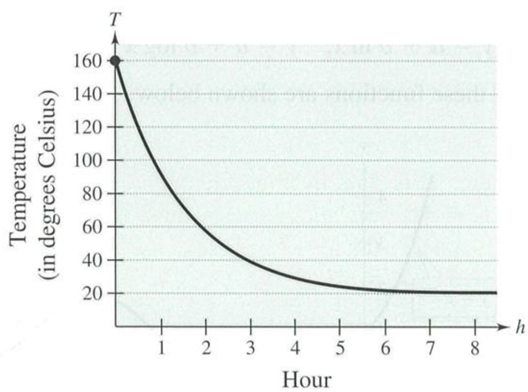
$$T = 20 + 140e^{-0.68h}.$$

DATA

Spreadsheet at LarsonPrecalculus.com

Hour, h	Temperature, T
0	160°
1	90°
2	56°
3	38°
4	29°
5	24°

(a) The figure below shows the graph of the model. Use the graph to identify the horizontal asymptote of the model and interpret the asymptote in the context of the problem.



(b) Use the model to approximate the time it took for the object to reach a temperature of 100°C.

Exploration

True or False? **In Exercises 89–92, rewrite each verbal statement as an equation. Then decide whether the statement is true or false. Justify your answer.**

89. The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.

90. The logarithm of the sum of two numbers is equal to the product of the logarithms of the numbers.

91. The logarithm of the difference of two numbers is equal to the difference of the logarithms of the numbers.

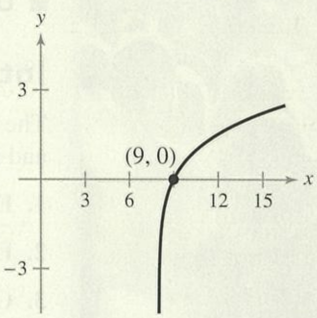
92. The logarithm of the quotient of two numbers is equal to the difference of the logarithms of the numbers.

93. Think About It Is it possible for a logarithmic equation to have more than one extraneous solution? Explain.

94. HOW DO YOU SEE IT? Solving $\log_3 x + \log_3(x - 8) = 2$ algebraically, the solutions appear to be $x = 9$ and $x = -1$. Use the graph of

$$y = \log_3 x + \log_3(x - 8) - 2$$

to determine whether each value is an actual solution of the equation. Explain.



95. Finance You are investing P dollars at an annual interest rate of r, compounded continuously, for t years. Which change below results in the highest value of the investment? Explain.

(a) Double the amount you invest.

(b) Double your interest rate.

(c) Double the number of years.

96. Think About It Are the times required for the investments in Exercises 71 and 72 to quadruple twice as long as the times for them to double? Give a reason for your answer and verify your answer algebraically.

97. Effective Yield The *effective yield* of an investment plan is the percent increase in the balance after 1 year. Find the effective yield for each investment plan. Which investment plan has the greatest effective yield? Which investment plan will have the highest balance after 5 years?

(a) 7% annual interest rate, compounded annually

(b) 7% annual interest rate, compounded continuously

(c) 7% annual interest rate, compounded quarterly

(d) 7.25% annual interest rate, compounded quarterly

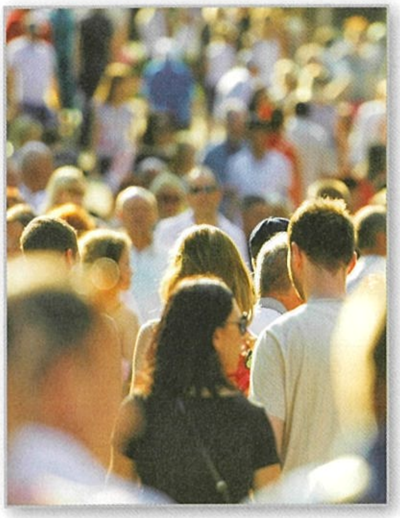
98. Graphical Reasoning Let $f(x) = \log_a x$ and $g(x) = a^x$, where $a > 1$.

(a) Let $a = 1.2$ and use a graphing utility to graph the two functions in the same viewing window. What do you observe? Approximate any points of intersection of the two graphs.

(b) Determine the value(s) of a for which the two graphs have one point of intersection.

(c) Determine the value(s) of a for which the two graphs have two points of intersection.

3.5 Exponential and Logarithmic Models



Exponential growth and decay models can often represent populations. For example, in Exercise 30 on page 244, you will use exponential growth and decay models to compare the populations of several countries.

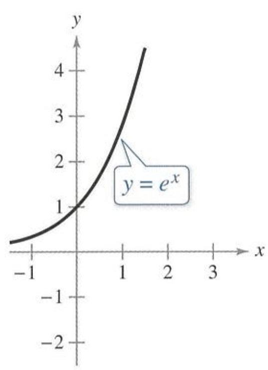
- **Recognize the five most common types of models involving exponential and logarithmic functions.**
- **Use exponential growth and decay functions to model and solve real-life problems.**
- **Use Gaussian functions to model and solve real-life problems.**
- **Use logistic growth functions to model and solve real-life problems.**
- **Use logarithmic functions to model and solve real-life problems.**

Introduction

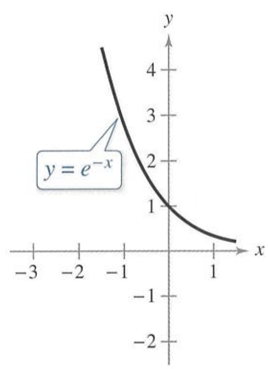
The five most common types of mathematical models involving exponential functions and logarithmic functions are listed below.

1. **Exponential growth model:** $y = ae^{bx}, \quad b > 0$
2. **Exponential decay model:** $y = ae^{-bx}, \quad b > 0$
3. **Gaussian model:** $y = ae^{-(x-b)^2/c}$
4. **Logistic growth model:** $y = \dfrac{a}{1 + be^{-rx}}$
5. **Logarithmic models:** $y = a + b \ln x, \quad y = a + b \log x$

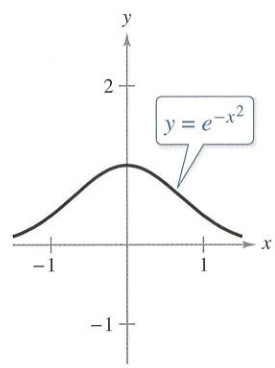
The basic shapes of the graphs of these functions are shown below.



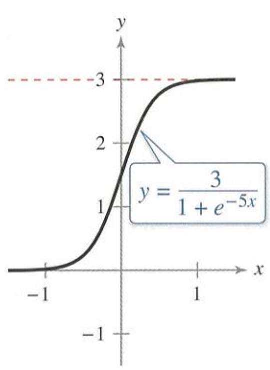
Exponential growth model



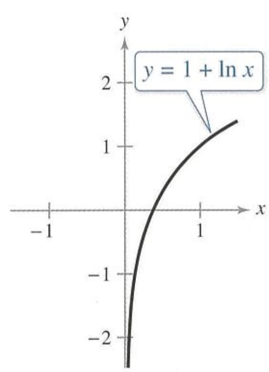
Exponential decay model



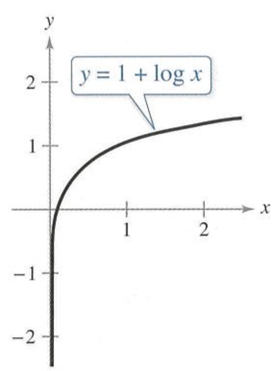
Gaussian model



Logistic growth model



Natural logarithmic model



Common logarithmic model

You often gain insight into a situation modeled by an exponential or logarithmic function by identifying and interpreting the asymptotes of the graph of the function. Identify the asymptote(s) of the graph of each function shown above.