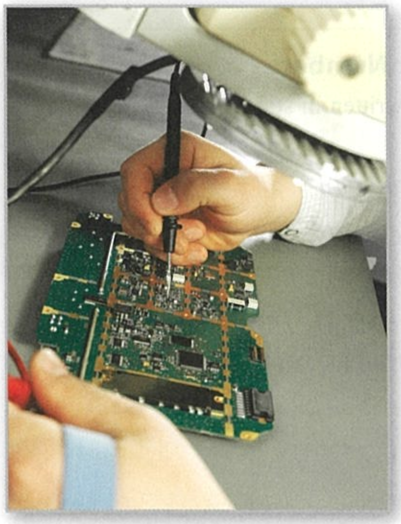
2.4 Complex Numbers



Complex numbers are often used in electrical engineering. For example, in Exercise 87 on page 151, you will use complex numbers to find the impedance of an electrical circuit.

- **Use the imaginary unit i to write complex numbers.**
- **Add, subtract, and multiply complex numbers.**
- **Use complex conjugates to write the quotient of two complex numbers in standard form.**
- **Find complex solutions of quadratic equations.**

The Imaginary Unit i

You have learned that some quadratic equations have no real solutions. For example, the quadratic equation

$$x^2 + 1 = 0$$

has no real solution because there is no real number x that can be squared to produce -1. To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit i,** defined as

$$i = \sqrt{-1} \qquad \text{Imaginary unit}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, you obtain the set of **complex numbers.** Each complex number can be written in the **standard form $a + bi$.** For example, the standard form of the complex number $-5 + \sqrt{-9}$ is $-5 + 3i$ because

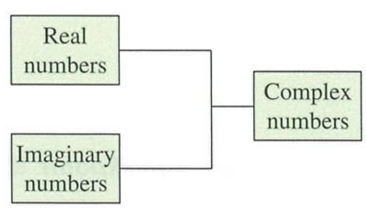
$$-5 + \sqrt{-9} = -5 + \sqrt{3^2(-1)} = -5 + 3\sqrt{-1} = -5 + 3i.$$

Definition of a Complex Number

Let a and b be real numbers. The number $a + bi$ is a **complex number** written in **standard form.** The real number a is the **real part** and the number bi (where b is a real number) is the **imaginary part** of the complex number.

When $b = 0$, the number $a + bi$ is a real number. When $b \neq 0$, the number $a + bi$ is an **imaginary number.** A number of the form bi, where $b \neq 0$, is a **pure imaginary number.**

Every real number a can be written as a complex number using $b = 0$. That is, for every real number a, $a = a + 0i$. So, the set of real numbers is a subset of the set of complex numbers, as shown in the figure below.



Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other

$$a + bi = c + di \qquad \text{Equality of two complex numbers}$$

if and only if $a = c$ and $b = d$.

Operations with Complex Numbers

To add (or subtract) two complex numbers, add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

For two complex numbers $a + bi$ and $c + di$ written in standard form, the sum and difference are

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i.$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number $a + bi$ is

$-(a + bi) = -a - bi.$ Additive inverse

So, you have $(a + bi) + (-a - bi) = 0 + 0i = 0.$

EXAMPLE 1 Adding and Subtracting Complex Numbers

a. $(4 + 7i) + (1 - 6i) = 4 + 7i + 1 - 6i$ Remove parentheses.

$= (4 + 1) + (7 - 6)i$ Group like terms.

$= 5 + i$ Write in standard form.

b. $(1 + 2i) + (3 - 2i) = 1 + 2i + 3 - 2i$ Remove parentheses.

$= (1 + 3) + (2 - 2)i$ Group like terms.

$= 4 + 0i$ Simplify.

$= 4$ Write in standard form.

REMARK Note that the sum of two complex numbers can be a real number.

c. $3i - (-2 + 3i) - (2 + 5i) = 3i + 2 - 3i - 2 - 5i$

$= (2 - 2) + (3 - 3 - 5)i$

$= 0 - 5i$

$= -5i$

d. $(3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i$

$= (3 + 4 - 7) + (2 - 1 - 1)i$

$= 0 + 0i$

$= 0$

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Perform each operation and write the result in standard form.

a. $(7 + 3i) + (5 - 4i)$

b. $(3 + 4i) - (5 - 3i)$

c. $2i + (-3 - 4i) - (-3 - 3i)$

d. $(5 - 3i) + (3 + 5i) - (8 + 2i)$

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Properties of Addition and Multiplication

Commutative Properties of Addition and Multiplication

Distributive Property of Multiplication Over Addition

Note the use of these properties when multiplying two complex numbers.

$(a + bi)(c + di) = a(c + di) + bi(c + di)$ Distributive Property

$= ac + (ad)i + (bc)i + (bd)i^2$ Distributive Property

$= ac + (ad)i + (bc)i + (bd)(-1)$ $i^2 = -1$

$= ac - bd + (ad)i + (bc)i$ Commutative Property

$= (ac - bd) + (ad + bc)i$ Associative Property

ALGEBRA HELP To review the FOIL method, see Appendix A.3.

The procedure shown above is similar to multiplying two binomials and combining like terms, as in the FOIL method. So, you do not need to memorize this procedure.

EXAMPLE 2 Multiplying Complex Numbers

See LarsonPrecalculus.com for an interactive version of this type of example.

a. $4(-2 + 3i) = 4(-2) + 4(3i)$ Distributive Property

$= -8 + 12i$ Simplify.

b. $(2 - i)(4 + 3i) = 8 + 6i - 4i - 3i^2$ FOIL Method

$= 8 + 6i - 4i - 3(-1)$ $i^2 = -1$

$= (8 + 3) + (6 - 4)i$ Group like terms.

$= 11 + 2i$ Write in standard form.

c. $(3 + 2i)(3 - 2i) = 9 - 6i + 6i - 4i^2$ FOIL Method

$= 9 - 6i + 6i - 4(-1)$ $i^2 = -1$

$= 9 + 4$ Simplify.

$= 13$ Write in standard form.

d. $(3 + 2i)^2 = (3 + 2i)(3 + 2i)$ Square of a binomial

$= 9 + 6i + 6i + 4i^2$ FOIL Method

$= 9 + 6i + 6i + 4(-1)$ $i^2 = -1$

$= 9 + 12i - 4$ Simplify.

$= 5 + 12i$ Write in standard form.

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Perform each operation and write the result in standard form.

a. $-5(3 - 2i)$

b. $(2 - 4i)(3 + 3i)$

c. $(4 + 5i)(4 - 5i)$

d. $(4 + 2i)^2$

Complex Conjugates

Notice in Example 2(c) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the form $a + bi$ and $a - bi$, called **complex conjugates.**

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2 \\ &= a^2 - b^2(-1) \\ &= a^2 + b^2\end{aligned}$$

REMARK Recall that the product of $a - b\sqrt{m}$ or $a + b\sqrt{m}$ and its conjugate is rational. Similarly, the product of a complex number and its conjugate is real.

EXAMPLE 3 Multiplying Conjugates

Multiply each complex number by its complex conjugate.

a. $1 + i$ **b.** $4 - 3i$

Solution

a. The complex conjugate of $1 + i$ is $1 - i$.

$$(1 + i)(1 - i) = 1^2 - i^2 = 1 - (-1) = 2$$

b. The complex conjugate of $4 - 3i$ is $4 + 3i$.

$$(4 - 3i)(4 + 3i) = 4^2 - (3i)^2 = 16 - 9i^2 = 16 - 9(-1) = 25$$

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Multiply each complex number by its complex conjugate.

a. $3 + 6i$ **b.** $2 - 5i$

To write the quotient of $a + bi$ and $c + di$ in standard form, where c and d are not both zero, multiply the numerator and denominator by the complex conjugate of the *denominator* to obtain

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right) = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)i.$$

REMARK Note that when you multiply a quotient of complex numbers by

$$\frac{c - di}{c - di}$$

you are multiplying the quotient by a form of 1. So, you are not changing the original expression, you are only writing an equivalent expression.

EXAMPLE 4 A Quotient of Complex Numbers in Standard Form

$$\frac{2 + 3i}{4 - 2i} = \frac{2 + 3i}{4 - 2i}\left(\frac{4 + 2i}{4 + 2i}\right)$$ Multiply numerator and denominator by complex conjugate of denominator.

$$= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2}$$ Expand.

$$= \frac{8 - 6 + 16i}{16 + 4}$$ $i^2 = -1$

$$= \frac{2 + 16i}{20}$$ Simplify.

$$= \frac{1}{10} + \frac{4}{5}i$$ Write in standard form.

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Write $\dfrac{2 + i}{2 - i}$ in standard form.

Complex Solutions of Quadratic Equations

You can write a number such as $\sqrt{-3}$ in standard form by factoring out $i = \sqrt{-1}$.

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}i$$

The number $\sqrt{3}i$ is the *principal square root* of -3.

REMARK The definition of principal square root uses the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

for $a > 0$ and $b < 0$. This rule is not valid when *both* a and b are negative. For example,

$$\begin{aligned}\sqrt{-5}\sqrt{-5} &= \sqrt{5(-1)}\sqrt{5(-1)}\\ &= \sqrt{5}i\sqrt{5}i\\ &= \sqrt{25i^2}\\ &= 5i^2\\ &= -5\end{aligned}$$

whereas

$$\sqrt{(-5)(-5)} = \sqrt{25} = 5.$$

Be sure to convert complex numbers to standard form *before* performing any operations.

Principal Square Root of a Negative Number

When a is a positive real number, the **principal square root** of $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}i.$$

EXAMPLE 5 Writing Complex Numbers in Standard Form

a. $\sqrt{-3}\sqrt{-12} = \sqrt{3}i\sqrt{12}i = \sqrt{36}i^2 = 6(-1) = -6$

b. $\sqrt{-48} - \sqrt{-27} = \sqrt{48}i - \sqrt{27}i = 4\sqrt{3}i - 3\sqrt{3}i = \sqrt{3}i$

c. $(-1 + \sqrt{-3})^2 = (-1 + \sqrt{3}i)^2 = 1 - 2\sqrt{3}i + 3(-1)$
$= -2 - 2\sqrt{3}i$

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Write $\sqrt{-14}\sqrt{-2}$ in standard form.

EXAMPLE 6 Complex Solutions of a Quadratic Equation

Solve $3x^2 - 2x + 5 = 0$.

Solution

$x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$ Quadratic Formula

$= \dfrac{2 \pm \sqrt{-56}}{6}$ Simplify.

$= \dfrac{2 \pm 2\sqrt{14}i}{6}$ Write $\sqrt{-56}$ in standard form.

$= \dfrac{1}{3} \pm \dfrac{\sqrt{14}}{3}i$ Write solution in standard form.

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Solve $8x^2 + 14x + 9 = 0$.

ALGEBRA HELP To review the Quadratic Formula, see Appendix A.5.

Summarize (Section 2.4)

1. Explain how to write complex numbers using the imaginary unit i *(page 145)*.
2. Explain how to add, subtract, and multiply complex numbers *(pages 146 and 147, Examples 1 and 2)*.
3. Explain how to use complex conjugates to write the quotient of two complex numbers in standard form *(page 148, Example 4)*.
4. Explain how to find complex solutions of a quadratic equation *(page 149, Example 6)*.

2.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. A ________ number has the form $a + bi$, where $a \neq 0$, $b = 0$.
2. An ________ number has the form $a + bi$, where $a \neq 0$, $b \neq 0$.
3. A ________ ________ number has the form $a + bi$, where $a = 0$, $b \neq 0$.
4. The imaginary unit i is defined as $i =$ ________, where $i^2 =$ ________.
5. When a is a positive real number, the ________ ________ root of $-a$ is defined as $\sqrt{-a} = \sqrt{a}i$.
6. The numbers $a + bi$ and $a - bi$ are called ________ ________, and their product is a real number $a^2 + b^2$.

Skills and Applications

Equality of Complex Numbers In Exercises 7–10, find real numbers a and b such that the equation is true.

7. $a + bi = 9 + 8i$
8. $a + bi = 10 - 5i$
9. $(a - 2) + (b + 1)i = 6 + 5i$
10. $(a + 2) + (b - 3)i = 4 + 7i$

Writing a Complex Number in Standard Form In Exercises 11–22, write the complex number in standard form.

11. $2 + \sqrt{-25}$
12. $4 + \sqrt{-49}$
13. $1 - \sqrt{-12}$
14. $2 - \sqrt{-18}$
15. $\sqrt{-40}$
16. $\sqrt{-27}$
17. 23
18. 50
19. $-6i + i^2$
20. $-2i^2 + 4i$
21. $\sqrt{-0.04}$
22. $\sqrt{-0.0025}$

Adding or Subtracting Complex Numbers In Exercises 23–30, perform the operation and write the result in standard form.

23. $(5 + i) + (2 + 3i)$
24. $(13 - 2i) + (-5 + 6i)$
25. $(9 - i) - (8 - i)$
26. $(3 + 2i) - (6 + 13i)$
27. $(-2 + \sqrt{-8}) + (5 - \sqrt{-50})$
28. $(8 + \sqrt{-18}) - (4 + 3\sqrt{2}i)$
29. $13i - (14 - 7i)$
30. $25 + (-10 + 11i) + 15i$

Multiplying Complex Numbers In Exercises 31–38, perform the operation and write the result in standard form.

31. $(1 + i)(3 - 2i)$
32. $(7 - 2i)(3 - 5i)$
33. $12i(1 - 9i)$
34. $-8i(9 + 4i)$
35. $(\sqrt{2} + 3i)(\sqrt{2} - 3i)$
36. $(4 + \sqrt{7}i)(4 - \sqrt{7}i)$
37. $(6 + 7i)^2$
38. $(5 - 4i)^2$

Multiplying Conjugates In Exercises 39–46, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.

39. $9 + 2i$
40. $8 - 10i$
41. $-1 - \sqrt{5}i$
42. $-3 + \sqrt{2}i$
43. $\sqrt{-20}$
44. $\sqrt{-15}$
45. $\sqrt{6}$
46. $1 + \sqrt{8}$

A Quotient of Complex Numbers in Standard Form In Exercises 47–54, write the quotient in standard form.

47. $\dfrac{2}{4 - 5i}$
48. $\dfrac{13}{1 - i}$
49. $\dfrac{5 + i}{5 - i}$
50. $\dfrac{6 - 7i}{1 - 2i}$
51. $\dfrac{9 - 4i}{i}$
52. $\dfrac{8 + 16i}{2i}$
53. $\dfrac{3i}{(4 - 5i)^2}$
54. $\dfrac{5i}{(2 + 3i)^2}$

Performing Operations with Complex Numbers In Exercises 55–58, perform the operation and write the result in standard form.

55. $\dfrac{2}{1 + i} - \dfrac{3}{1 - i}$
56. $\dfrac{2i}{2 + i} + \dfrac{5}{2 - i}$
57. $\dfrac{i}{3 - 2i} + \dfrac{2i}{3 + 8i}$
58. $\dfrac{1 + i}{i} - \dfrac{3}{4 - i}$

Writing a Complex Number in Standard Form In Exercises 59–66, write the complex number in standard form.

59. $\sqrt{-6}\sqrt{-2}$
60. $\sqrt{-5}\sqrt{-10}$
61. $(\sqrt{-15})^2$
62. $(\sqrt{-75})^2$
63. $\sqrt{-8} + \sqrt{-50}$
64. $\sqrt{-45} - \sqrt{-5}$
65. $(3 + \sqrt{-5})(7 - \sqrt{-10})$
66. $(2 - \sqrt{-6})^2$

Complex Solutions of a Quadratic Equation In Exercises 67–76, use the Quadratic Formula to solve the quadratic equation.

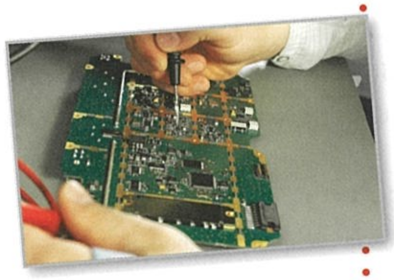
67. $x^2 - 2x + 2 = 0$
68. $x^2 + 6x + 10 = 0$
69. $4x^2 + 16x + 17 = 0$
70. $9x^2 - 6x + 37 = 0$
71. $4x^2 + 16x + 21 = 0$
72. $16t^2 - 4t + 3 = 0$
73. $\frac{3}{2}x^2 - 6x + 9 = 0$
74. $\frac{7}{8}x^2 - \frac{3}{4}x + \frac{5}{16} = 0$
75. $1.4x^2 - 2x + 10 = 0$
76. $4.5x^2 - 3x + 12 = 0$

Simplifying a Complex Number In Exercises 77–86, simplify the complex number and write it in standard form.

77. $-6i^3 + i^2$
78. $4i^2 - 2i^3$
79. $-14i^5$
80. $(-i)^3$
81. $(\sqrt{-72})^3$
82. $(\sqrt{-2})^6$
83. $\dfrac{1}{i^3}$
84. $\dfrac{1}{(2i)^3}$
85. $(3i)^4$
86. $(-i)^6$

87. **Impedance of a Circuit** The opposition to current in an electrical circuit is called its impedance. The impedance z in a parallel circuit with two pathways satisfies the equation

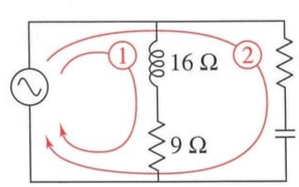
$$\frac{1}{z} = \frac{1}{z_1} + \frac{1}{z_2}$$



where z_1 is the impedance (in ohms) of pathway 1 and z_2 is the impedance (in ohms) of pathway 2.

(a) The impedance of each pathway in a parallel circuit is found by adding the impedances of all components in the pathway. Use the table to find z_1 and z_2

	Resistor	Inductor	Capacitor
Symbol	$a\ \Omega$	$b\ \Omega$	$c\ \Omega$
Impedance	a	bi	$-ci$



(b) Find the impedance z.

88. **Cube of a Complex Number** Cube each complex number.
(a) $-1 + \sqrt{3}i$ (b) $-1 - \sqrt{3}i$

Exploration

True or False? In Exercises 89–92, determine whether the statement is true or false. Justify your answer.

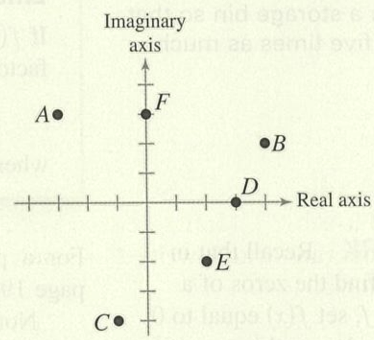
89. The sum of two complex numbers is always a real number.
90. There is no complex number that is equal to its complex conjugate.
91. $-i\sqrt{6}$ is a solution of $x^4 - x^2 + 14 = 56$.
92. $i^{44} + i^{150} - i^{74} - i^{109} + i^{61} = -1$

93. **Pattern Recognition** Find the missing values.

$i^1 = i$	$i^2 = -1$	$i^3 = -i$	$i^4 = 1$
$i^5 = \square$	$i^6 = \square$	$i^7 = \square$	$i^8 = \square$
$i^9 = \square$	$i^{10} = \square$	$i^{11} = \square$	$i^{12} = \square$

What pattern do you see? Write a brief description of how you would find i raised to any positive integer power.

94. **HOW DO YOU SEE IT?** The coordinate system shown below is called the complex plane. In the complex plane, the point (a, b) corresponds to the complex number $a + bi$.



Match each complex number with its corresponding point.
(i) 3 (ii) $3i$ (iii) $4 + 2i$
(iv) $2 - 2i$ (v) $-3 + 3i$ (vi) $-1 - 4i$

95. **Error Analysis** Describe the error.

$$\sqrt{-6}\sqrt{-6} = \sqrt{(-6)(-6)} = \sqrt{36} = 6 \quad ✗$$

96. **Proof** Prove that the complex conjugate of the product of two complex numbers $a_1 + b_1i$ and $a_2 + b_2i$ is the product of their complex conjugates.

97. **Proof** Prove that the complex conjugate of the sum of two complex numbers $a_1 + b_1i$ and $a_2 + b_2i$ is the sum of their complex conjugates.

2.5 Zeros of Polynomial Functions

Finding zeros of polynomial functions is an important part of solving many real-life problems. For example, in Exercise 105 on page 164, you will use the zeros of a polynomial function to redesign a storage bin so that it can hold five times as much food.

- **Use the Fundamental Theorem of Algebra to determine numbers of zeros of polynomial functions.**
- **Find rational zeros of polynomial functions.**
- **Find complex zeros using conjugate pairs.**
- **Find zeros of polynomials by factoring.**
- **Use Descartes's Rule of Signs and the Upper and Lower Bound Rules to find zeros of polynomials.**
- **Find zeros of polynomials in real-life applications.**

The Fundamental Theorem of Algebra

In the complex number system, every nth-degree polynomial function has *precisely* n zeros. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by German mathematician Carl Friedrich Gauss (1777–1855).

> **The Fundamental Theorem of Algebra**
>
> If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the **Linear Factorization Theorem.**

> **Linear Factorization Theorem**
>
> If $f(x)$ is a polynomial of degree n, where $n > 0$, then $f(x)$ has precisely n linear factors
>
> $$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$
>
> where $c_1, c_2, . . . , c_n$ are complex numbers.

For a proof of the Linear Factorization Theorem, see Proofs in Mathematics on page 194.

Note that the Fundamental Theorem of Algebra and the Linear Factorization Theorem tell you only that the zeros or factors of a polynomial exist, not how to find them. Such theorems are called **existence theorems.**

REMARK Recall that in order to find the zeros of a function f, set $f(x)$ equal to 0 and solve the resulting equation for x. For instance, the function in Example 1(a) has a zero at $x = 2$ because

$$x - 2 = 0$$
$$x = 2.$$

EXAMPLE 1 Zeros of Polynomial Functions

See LarsonPrecalculus.com for an interactive version of this type of example.

a. The first-degree polynomial function $f(x) = x - 2$ has exactly *one* zero: $x = 2$.

b. The second-degree polynomial function $f(x) = x^2 - 6x + 9 = (x - 3)(x - 3)$ has exactly *two* zeros: $x = 3$ and $x = 3$ (a *repeated zero*).

c. The third-degree polynomial function $f(x) = x^3 + 4x = x(x - 2i)(x + 2i)$ has exactly *three* zeros: $x = 0$, $x = 2i$, and $x = -2i$.

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Determine the number of zeros of the polynomial function $f(x) = x^4 - 1$.

The Rational Zero Test

The **Rational Zero Test** relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

Although they were not contemporaries, French mathematician Jean Le Rond d'Alembert (1717–1783) worked independently of Carl Friedrich Gauss in trying to prove the Fundamental Theorem of Algebra. His efforts were such that, in France, the Fundamental Theorem of Algebra is frequently known as d'Alembert's Theorem.

The Rational Zero Test

If the polynomial

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

has *integer* coefficients, then every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, and

$p =$ a factor of the constant term a_0

$q =$ a factor of the leading coefficient a_n.

To use the Rational Zero Test, you should first list all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.

$$\text{Possible rational zeros: } \frac{\text{Factors of constant term}}{\text{Factors of leading coefficient}}$$

Having formed this list of *possible rational zeros,* use a trial-and-error method to determine which, if any, are actual zeros of the polynomial. Note that when the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

EXAMPLE 2 Rational Zero Test with Leading Coefficient of 1

Find (if possible) the rational zeros of

$$f(x) = x^3 + x + 1.$$

Solution The leading coefficient is 1, so the possible rational zeros are the factors of the constant term.

Possible rational zeros: 1 and -1

Testing these possible zeros shows that neither works.

$$f(1) = (1)^3 + 1 + 1$$
$$= 3$$
$$f(-1) = (-1)^3 + (-1) + 1$$
$$= -1$$

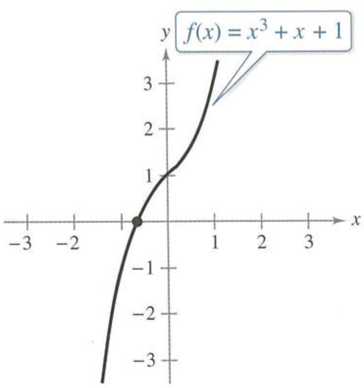

Figure 2.15

So, the given polynomial has *no* rational zeros. Note from the graph of f in Figure 2.15 that f does have one real zero between -1 and 0. However, by the Rational Zero Test, you know that this real zero is *not* a rational number.

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Find (if possible) the rational zeros of

$$f(x) = x^3 + 2x^2 + 6x - 4.$$

EXAMPLE 3 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of

$$f(x) = x^4 - x^3 + x^2 - 3x - 6.$$

Solution The leading coefficient is 1, so the possible rational zeros are the factors of the constant term.

Possible rational zeros: $\pm1, \pm2, \pm3, \pm6$

By applying synthetic division successively, you find that $x = -1$ and $x = 2$ are the only two rational zeros.

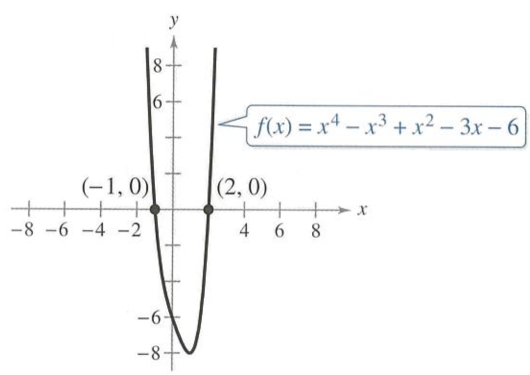
-1	1	-1	1	-3	-6	
		-1	2	-3	6	
	1	-2	3	-6	0	→ 0 remainder, so $x = -1$ is a zero.

2	1	-2	3	-6	
		2	0	6	
	1	0	3	0	→ 0 remainder, so $x = 2$ is a zero.

So, $f(x)$ factors as

$$f(x) = (x + 1)(x - 2)(x^2 + 3).$$

The factor $(x^2 + 3)$ produces no real zeros, so $x = -1$ and $x = 2$ are the only *real* zeros of f. The figure below verifies this.



REMARK When there are few possible rational zeros, as in Example 2, it may be quicker to test the zeros by evaluating the function. When there are more possible rational zeros, as in Example 3, it may be quicker to use a different approach to test the zeros, such as using synthetic division or sketching a graph.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Find the rational zeros of

$$f(x) = x^3 - 15x^2 + 75x - 125.$$

When the leading coefficient of a polynomial is not 1, the number of possible rational zeros can increase dramatically. In such cases, the search can be shortened in several ways.

1. A graphing utility can help to speed up the calculations.
2. A graph can give good estimates of the locations of the zeros.
3. The Intermediate Value Theorem, along with a table of values, can give approximations of the zeros.
4. Synthetic division can be used to test the possible rational zeros.

After finding the first zero, the search becomes simpler by working with the lower-degree polynomial obtained in synthetic division, as shown in Example 3.

EXAMPLE 4 Using the Rational Zero Test

Find the rational zeros of $f(x) = 2x^3 + 3x^2 - 8x + 3$.

Solution The leading coefficient is 2 and the constant term is 3.

$$\textit{Possible rational zeros: } \frac{\text{Factors of 3}}{\text{Factors of 2}} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}$$

REMARK Remember that when you find the rational zeros of a polynomial function with many possible rational zeros, as in Example 4, you must use trial and error. There is no quick algebraic method to determine which of the possibilities is an actual zero; however, sketching a graph may be helpful.

By synthetic division, $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$f(x) = (x - 1)(2x^2 + 5x - 3)$$
$$= (x - 1)(2x - 1)(x + 3)$$

which shows that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$.

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Find the rational zeros of

$$f(x) = 2x^3 + x^2 - 13x + 6.$$

Recall from Section 2.2 that if $x = a$ is a zero of the polynomial function f, then $x = a$ is a solution of the polynomial equation $f(x) = 0$.

EXAMPLE 5 Solving a Polynomial Equation

Find all real solutions of $-10x^3 + 15x^2 + 16x - 12 = 0$.

Solution The leading coefficient is -10 and the constant term is -12.

$$\textit{Possible rational solutions: } \frac{\text{Factors of } -12}{\text{Factors of } -10} = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 5, \pm 10}$$

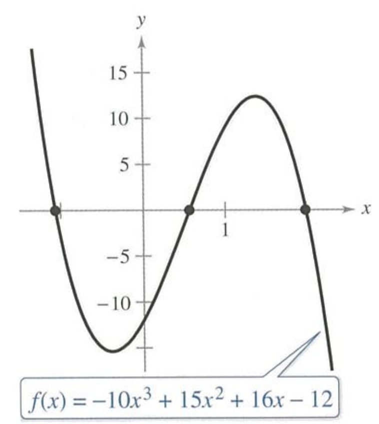

Figure 2.16

With so many possibilities (32, in fact), it is worth your time to sketch a graph. In Figure 2.16, three reasonable solutions appear to be $x = -\frac{6}{5}$, $x = \frac{1}{2}$, and $x = 2$. Testing these by synthetic division shows that $x = 2$ is the only rational solution. So, you have

$$(x - 2)(-10x^2 - 5x + 6) = 0.$$

Using the Quadratic Formula to solve $-10x^2 - 5x + 6 = 0$, you find that the two additional solutions are irrational numbers.

$$x = \frac{5 + \sqrt{265}}{-20} \approx -1.0639$$

and

$$x = \frac{5 - \sqrt{265}}{-20} \approx 0.5639$$

ALGEBRA HELP To review the Quadratic Formula, see Appendix A.5.

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Find all real solutions of

$$-2x^3 - 5x^2 + 15x + 18 = 0.$$

Conjugate Pairs

In Example 1(c), note that the two complex zeros $2i$ and $-2i$ are complex conjugates. That is, they are of the forms $a + bi$ and $a - bi$.

Complex Zeros Occur in Conjugate Pairs

Let f be a polynomial function that has *real coefficients.* If $a + bi$, where $b \neq 0$, is a zero of the function, then the complex conjugate $a - bi$ is also a zero of the function.

Be sure you see that this result is true only when the polynomial function has *real coefficients.* For example, the result applies to the function $f(x) = x^2 + 1$, but not to the function $g(x) = x - i$.

EXAMPLE 6 Finding a Polynomial Function with Given Zeros

Find a fourth-degree polynomial function f with real coefficients that has -1, -1, and $3i$ as zeros.

Solution You are given that $3i$ is a zero of f *and* the polynomial has real coefficients, so you know that the complex conjugate $-3i$ must also be a zero. Using the Linear Factorization Theorem, write $f(x)$ as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

For simplicity, let $a = 1$ to obtain

$$f(x) = (x^2 + 2x + 1)(x^2 + 9) = x^4 + 2x^3 + 10x^2 + 18x + 9.$$

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Find a fourth-degree polynomial function f with real coefficients that has 2, -2, and $-7i$ as zeros.

EXAMPLE 7 Finding a Polynomial Function with Given Zeros

Find the cubic polynomial function f with real coefficients that has 2 and $1 - i$ as zeros, and $f(1) = 3$.

Solution You are given that $1 - i$ is a zero of f, so the complex conjugate $1 + i$ is also a zero.

$$\begin{aligned} f(x) &= a(x - 2)[x - (1 - i)][x - (1 + i)] \\ &= a(x - 2)[(x - 1) + i][(x - 1) - i] \\ &= a(x - 2)[(x - 1)^2 + 1] \\ &= a(x - 2)(x^2 - 2x + 2) \\ &= a(x^3 - 4x^2 + 6x - 4) \end{aligned}$$

To find the value of a, use the fact that $f(1) = 3$ to obtain

$$a[(1)^3 - 4(1)^2 + 6(1) - 4] = 3.$$

So, $a = -3$ and

$$f(x) = -3(x^3 - 4x^2 + 6x - 4) = -3x^3 + 12x^2 - 18x + 12.$$

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Find the *quartic* (fourth-degree) polynomial function f with real coefficients that has 1, -2, and $2i$ as zeros, and $f(-1) = 10$.

Factoring a Polynomial

The Linear Factorization Theorem states that you can write any nth-degree polynomial as the product of n linear factors.

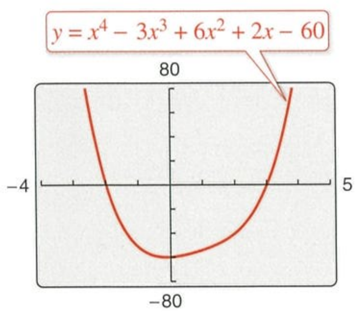
$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n)$$

This result includes the possibility that some of the values of c_i are imaginary. The theorem below states that you can write $f(x)$ as the product of linear and quadratic factors with real coefficients. For a proof of this theorem, see Proofs in Mathematics on page 194.

> **Factors of a Polynomial**
>
> Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

A quadratic factor with no real zeros is *prime* or **irreducible over the reals.** Note that this is not the same as being *irreducible over the rationals.* For example, the quadratic $x^2 + 1 = (x - i)(x + i)$ is irreducible over the reals (and therefore over the rationals). On the other hand, the quadratic $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$ is irreducible over the rationals but *reducible* over the reals.

TECHNOLOGY Another way to find the real zeros of the function in Example 8 is to use a graphing utility to graph the function (see figure).



Then use the *zero* or *root* feature of the graphing utility to determine that $x = -2$ and $x = 3$ are the real zeros.

EXAMPLE 8 Finding the Zeros of a Polynomial Function

Find all the zeros of $f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$ given that $1 + 3i$ is a zero of f.

Solution Complex zeros occur in conjugate pairs, so you know that $1 - 3i$ is also a zero of f. This means that both $[x - (1 + 3i)]$ and $[x - (1 - 3i)]$ are factors of $f(x)$. Multiplying these two factors produces

$$\begin{aligned}[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10.\end{aligned}$$

Using long division, divide $x^2 - 2x + 10$ into $f(x)$.

$$\begin{array}{r}
x^2 - x - 6 \\
x^2 - 2x + 10 \overline{\smash{)}\, x^4 - 3x^3 + 6x^2 + 2x - 60} \\
\underline{x^4 - 2x^3 + 10x^2} \\
-x^3 - 4x^2 + 2x \\
\underline{-x^3 + 2x^2 - 10x} \\
-6x^2 + 12x - 60 \\
\underline{-6x^2 + 12x - 60} \\
0
\end{array}$$

So, you have

$$f(x) = (x^2 - 2x + 10)(x^2 - x - 6) = (x^2 - 2x + 10)(x - 3)(x + 2)$$

and can conclude that the zeros of f are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

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Find all the zeros of $f(x) = 3x^3 - 2x^2 + 48x - 32$ given that $4i$ is a zero of f.

ALGEBRA HELP To review the techniques for polynomial long division, see Section 2.3.

In Example 8, without knowing that $1 + 3i$ is a zero of f, it is still possible to find all the zeros of the function. You can first use synthetic division to find the real zeros -2 and 3. Then, factor the polynomial as

$$(x + 2)(x - 3)(x^2 - 2x + 10).$$

Finally, use the Quadratic Formula to solve $x^2 - 2x + 10 = 0$ to obtain the zeros $1 + 3i$ and $1 - 3i$.

In Example 9, you will find all the zeros, including the imaginary zeros, of a fifth-degree polynomial function.

EXAMPLE 9 Finding the Zeros of a Polynomial Function

Write

$$f(x) = x^5 + x^3 + 2x^2 - 12x + 8$$

as the product of linear factors and list all the zeros of the function.

Solution The leading coefficient is 1, so the possible rational zeros are the factors of the constant term.

Possible rational zeros: $\pm 1, \pm 2, \pm 4$, and ± 8

By synthetic division, $x = 1$ and $x = -2$ are zeros.

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 1 & 2 & -12 & 8 \\ & & 1 & 1 & 2 & 4 & -8 \\ \hline & 1 & 1 & 2 & 4 & -8 & 0 \end{array} \longrightarrow \text{1 is a zero.}$$

$$\begin{array}{r|rrrrr} -2 & 1 & 1 & 2 & 4 & -8 \\ & & -2 & 2 & -8 & 8 \\ \hline & 1 & -1 & 4 & -4 & 0 \end{array} \longrightarrow \text{-2 is a zero.}$$

So, you have

$$f(x) = x^5 + x^3 + 2x^2 - 12x + 8$$
$$= (x - 1)(x + 2)(x^3 - x^2 + 4x - 4).$$

Factoring by grouping,

$$x^3 - x^2 + 4x - 4 = (x - 1)(x^2 + 4)$$

and by factoring $x^2 + 4$ as

$$x^2 + 4 = (x - 2i)(x + 2i)$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i)$$

which gives all five zeros of f.

$$x = 1, \quad x = 1, \quad x = -2, \quad x = 2i, \quad \text{and} \quad x = -2i$$

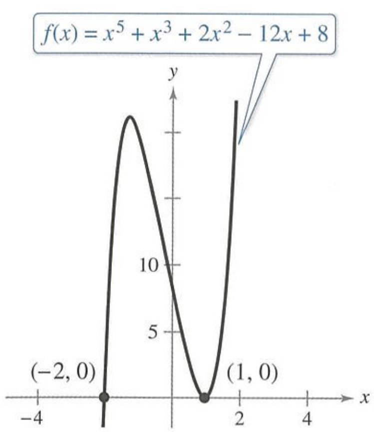

Figure 2.17

Figure 2.17 shows the graph of f. Notice that the *real* zeros are the only ones that appear as x-intercepts and that the real zero $x = 1$ is repeated.

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Write

$$f(x) = x^4 + 8x^2 - 9$$

as the product of linear factors and list all the zeros of the function.

Other Tests for Zeros of Polynomials

You know that an nth-degree polynomial function can have *at most n* real zeros. Of course, many nth-degree polynomial functions do not have that many real zeros. For example, $f(x) = x^2 + 1$ has no real zeros, and $f(x) = x^3 + 1$ has only one real zero. The theorem below, called **Descartes's Rule of Signs,** uses variations in sign to analyze the number of real zeros of a polynomial. A **variation in sign** means that two consecutive nonzero coefficients have opposite signs.

Descartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

1. The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer.
2. The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

When using Descartes's Rule of Signs, count a zero of multiplicity k as k zeros. For example, the polynomial $x^3 - 3x + 2$ has two variations in sign, and so it has either two positive or no positive real zeros. This polynomial factors as

$$x^3 - 3x + 2 = (x - 1)(x - 1)(x + 2)$$

so the two positive real zeros are $x = 1$ of multiplicity 2.

EXAMPLE 10 Using Descartes's Rule of Signs

Determine the possible numbers of positive and negative real zeros of

$$f(x) = 3x^3 - 5x^2 + 6x - 4.$$

Solution The original polynomial has *three* variations in sign.

+ to − + to −

$$f(x) = 3x^3 - 5x^2 + 6x - 4$$

− to +

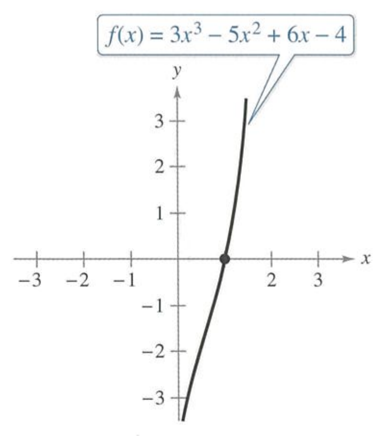

Figure 2.18

The polynomial

$$f(-x) = 3(-x)^3 - 5(-x)^2 + 6(-x) - 4$$
$$= -3x^3 - 5x^2 - 6x - 4$$

has no variations in sign. So, from Descartes's Rule of Signs, the polynomial

$$f(x) = 3x^3 - 5x^2 + 6x - 4$$

has either three positive real zeros or one positive real zero, and has no negative real zeros. Figure 2.18 shows that the function has only one real zero, $x = 1$.

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Determine the possible numbers of positive and negative real zeros of

$$f(x) = 2x^3 + 5x^2 + x + 8.$$

Another test for zeros of a polynomial function is related to the sign pattern in the last row of the synthetic division array. This test can give you an upper or lower bound for the real zeros of f. A real number c is an **upper bound** for the real zeros of f when no zeros are greater than c. Similarly, c is a **lower bound** when no real zeros of f are less than c.

Upper and Lower Bound Rules

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Divide $f(x)$ by $x - c$ using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, then c is an **upper bound** for the real zeros of f.
2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), then c is a **lower bound** for the real zeros of f.

EXAMPLE 11 Finding Real Zeros of a Polynomial Function

Find all real zeros of

$$f(x) = 6x^3 - 4x^2 + 3x - 2.$$

Solution List the possible rational zeros of f.

$$\frac{\text{Factors of } -2}{\text{Factors of } 6} = \frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6} = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{3}, \pm 2$$

The original polynomial $f(x)$ has three variations in sign. The polynomial

$$\begin{aligned} f(-x) &= 6(-x)^3 - 4(-x)^2 + 3(-x) - 2 \\ &= -6x^3 - 4x^2 - 3x - 2 \end{aligned}$$

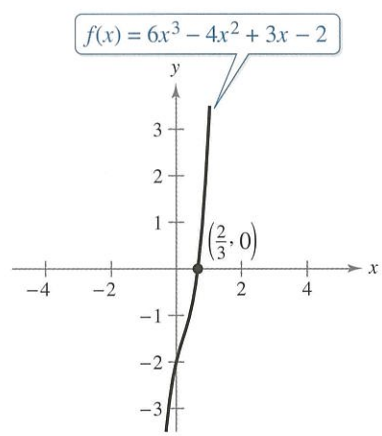
has no variations in sign. So, by Descartes's Rule of Signs, there are three positive real zeros or one positive real zero, and no negative real zeros. Test $x = 1$.

$$\begin{array}{r|rrrr} 1 & 6 & -4 & 3 & -2 \\ & & 6 & 2 & 5 \\ \hline & 6 & 2 & 5 & 3 \end{array}$$

This shows that $x = 1$ is not a zero. However, the last row has all positive entries, telling you that $x = 1$ is an upper bound for the real zeros. So, restrict the search to zeros between 0 and 1. By trial and error, $x = \frac{2}{3}$ is a zero, and factoring,

$$f(x) = \left(x - \frac{2}{3}\right)(6x^2 + 3).$$

The factor $6x^2 + 3$ has no real zeros, so it follows that $x = \frac{2}{3}$ is the only real zero, as verified in the graph of f at the right.



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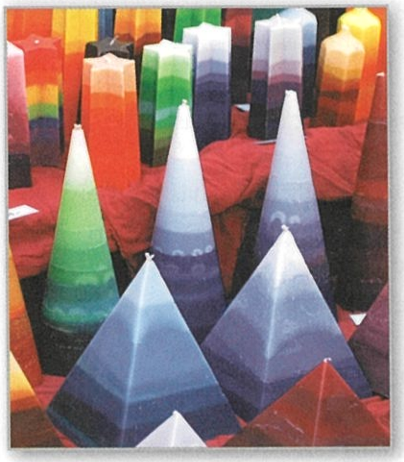
Find all real zeros of $f(x) = 8x^3 - 4x^2 + 6x - 3$.

Application



EXAMPLE 12 Using a Polynomial Model

You design candle making kits. Each kit contains 25 cubic inches of candle wax and a mold for making a pyramid-shaped candle. You want the height of the candle to be 2 inches less than the length of each side of the candle's square base. What should the dimensions of your candle mold be?

Solution The volume of a pyramid is $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height. The area of the base is x^2 and the height is $(x - 2)$. So, the volume of the pyramid is $V = \frac{1}{3}x^2(x - 2)$. Substitute 25 for the volume and solve for x.

$$25 = \frac{1}{3}x^2(x - 2) \qquad \text{Substitute 25 for } V.$$

$$75 = x^3 - 2x^2 \qquad \text{Multiply each side by 3, and distribute } x^2.$$

$$0 = x^3 - 2x^2 - 75 \qquad \text{Write in general form.}$$

The possible rational solutions are $x = \pm1, \pm3, \pm5, \pm15, \pm25, \pm75$. Note that in this case it makes sense to consider only positive x-values. Use synthetic division to test some of the possible solutions and determine that $x = 5$ is a solution.

$$\begin{array}{r|rrrr} 5 & 1 & -2 & 0 & -75 \\ & & 5 & 15 & 75 \\ \hline & 1 & 3 & 15 & 0 \end{array}$$

The other two solutions, which satisfy $x^2 + 3x + 15 = 0$, are imaginary, so discard them and conclude that the base of the candle mold should be 5 inches by 5 inches and the height should be $5 - 2 = 3$ inches.

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Rework Example 12 when each kit contains 147 cubic inches of candle wax and you want the height of the pyramid-shaped candle to be 2 inches more than the length of each side of the candle's square base.

Before concluding this section, here is an additional hint that can help you find the zeros of a polynomial function. When the terms of $f(x)$ have a common monomial factor, you should factor it out before applying the tests in this section. For example, writing $f(x) = x^4 - 5x^3 + 3x^2 + x = x(x^3 - 5x^2 + 3x + 1)$ shows that $x = 0$ is a zero of f. Obtain the remaining zeros by analyzing the cubic factor.

Summarize (Section 2.5)

1. State the Fundamental Theorem of Algebra and the Linear Factorization Theorem *(page 152, Example 1)*.
2. Explain how to use the Rational Zero Test *(page 153, Examples 2–5)*.
3. Explain how to use complex conjugates when analyzing a polynomial function *(page 156, Examples 6 and 7)*.
4. Explain how to find the zeros of a polynomial function *(page 157, Examples 8 and 9)*.
5. State Descartes's Rule of Signs and the Upper and Lower Bound Rules *(pages 159 and 160, Examples 10 and 11)*.
6. Describe a real-life application of finding the zeros of a polynomial function *(page 161, Example 12)*.

2.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. The ________ ________ of ________ states that if $f(x)$ is a polynomial of degree n $(n > 0)$, then f has at least one zero in the complex number system.
2. The ________ ________ ________ states that if $f(x)$ is a polynomial of degree n $(n > 0)$, then $f(x)$ has precisely n linear factors, $f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$, where $c_1, c_2, \ldots, c_n$ are complex numbers.
3. The test that gives a list of the possible rational zeros of a polynomial function is the ________ ________ Test.
4. If $a + bi$, where $b \neq 0$, is a complex zero of a polynomial with real coefficients, then so is its ________ ________, $a - bi$.
5. Every polynomial of degree $n > 0$ with real coefficients can be written as the product of ________ and ________ factors with real coefficients, where the ________ factors have no real zeros.
6. A quadratic factor that cannot be factored further as a product of linear factors containing real numbers is ________ over the ________.
7. The theorem that can be used to determine the possible numbers of positive and negative real zeros of a function is called ________ ________ of ________.
8. A real number c is a ________ bound for the real zeros of f when no real zeros are less than c, and is a ________ bound when no real zeros are greater than c.

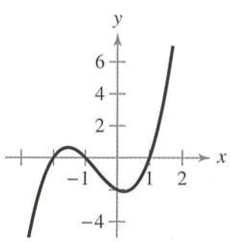
Skills and Applications

Zeros of Polynomial Functions **In Exercises 9–14, determine the number of zeros of the polynomial function.**

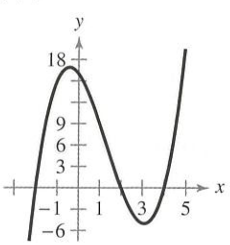
9. $f(x) = x^3 + 2x^2 + 1$
10. $f(x) = x^4 - 3x$
11. $g(x) = x^4 - x^5$
12. $f(x) = x^3 - x^6$
13. $f(x) = (x + 5)^2$
14. $h(t) = (t - 1)^2 - (t + 1)^2$

Using the Rational Zero Test **In Exercises 15–18, use the Rational Zero Test to list the possible rational zeros of f. Verify that the zeros of f shown in the graph are contained in the list.**

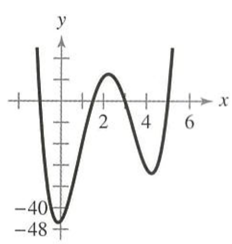
15. $f(x) = x^3 + 2x^2 - x - 2$



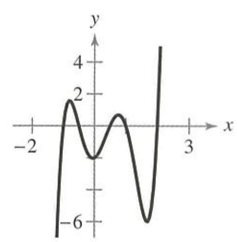
16. $f(x) = x^3 - 4x^2 - 4x + 16$



17. $f(x) = 2x^4 - 17x^3 + 35x^2 + 9x - 45$



18. $f(x) = 4x^5 - 8x^4 - 5x^3 + 10x^2 + x - 2$



Using the Rational Zero Test **In Exercises 19–28, find (if possible) the rational zeros of the function.**

19. $f(x) = x^3 - 7x - 6$
20. $f(x) = x^3 - 13x + 12$
21. $g(t) = t^3 - 4t^2 + 4$
22. $h(x) = x^3 - 19x + 30$
23. $h(t) = t^3 + 8t^2 + 13t + 6$
24. $g(x) = x^3 + 8x^2 + 12x + 18$
25. $C(x) = 2x^3 + 3x^2 - 1$
26. $f(x) = 3x^3 - 19x^2 + 33x - 9$
27. $g(x) = 9x^4 - 9x^3 - 58x^2 + 4x + 24$
28. $f(x) = 2x^4 - 15x^3 + 23x^2 + 15x - 25$

Solving a Polynomial Equation In Exercises 29–32, find all real solutions of the polynomial equation.

29. $-5x^3 + 11x^2 - 4x - 2 = 0$
30. $8x^3 + 10x^2 - 15x - 6 = 0$
31. $x^4 + 6x^3 + 3x^2 - 16x + 6 = 0$
32. $x^4 + 8x^3 + 14x^2 - 17x - 42 = 0$

Using the Rational Zero Test In Exercises 33–36, (a) list the possible rational zeros of f, (b) sketch the graph of f so that some of the possible zeros in part (a) can be disregarded, and then (c) determine all real zeros of f.

33. $f(x) = x^3 + x^2 - 4x - 4$
34. $f(x) = -3x^3 + 20x^2 - 36x + 16$
35. $f(x) = -4x^3 + 15x^2 - 8x - 3$
36. $f(x) = 4x^3 - 12x^2 - x + 15$

Using the Rational Zero Test In Exercises 37–40, (a) list the possible rational zeros of f, (b) use a graphing utility to graph f so that some of the possible zeros in part (a) can be disregarded, and then (c) determine all real zeros of f.

37. $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$
38. $f(x) = 4x^4 - 17x^2 + 4$
39. $f(x) = 32x^3 - 52x^2 + 17x + 3$
40. $f(x) = 4x^3 + 7x^2 - 11x - 18$

Finding a Polynomial Function with Given Zeros In Exercises 41–46, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)

41. $1, 5i$
42. $4, -3i$
43. $2, 2, 1 + i$
44. $-1, 5, 3 - 2i$
45. $\frac{2}{3}, -1, 3 + \sqrt{2}i$
46. $-\frac{5}{2}, -5, 1 + \sqrt{3}i$

Finding a Polynomial Function with Given Zeros In Exercises 47–50, find the polynomial function f with real coefficients that has the given degree, zeros, and solution point.

	Degree	Zeros	Solution Point
47.	4	$-2, 1, i$	$f(0) = -4$
48.	4	$-1, 2, \sqrt{2}i$	$f(1) = 12$
49.	3	$-3, 1 + \sqrt{3}i$	$f(-2) = 12$
50.	3	$-2, 1 - \sqrt{2}i$	$f(-1) = -12$

Factoring a Polynomial In Exercises 51–54, write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.

51. $f(x) = x^4 + 2x^2 - 8$
52. $f(x) = x^4 + 6x^2 - 27$
53. $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$
(*Hint:* One factor is $x^2 - 6$.)
54. $f(x) = x^4 - 3x^3 - x^2 - 12x - 20$
(*Hint:* One factor is $x^2 + 4$.)

Finding the Zeros of a Polynomial Function In Exercises 55–60, use the given zero to find all the zeros of the function.

	Function	Zero
55.	$f(x) = x^3 - x^2 + 4x - 4$	$2i$
56.	$f(x) = 2x^3 + 3x^2 + 18x + 27$	$3i$
57.	$g(x) = x^3 - 8x^2 + 25x - 26$	$3 + 2i$
58.	$g(x) = x^3 + 9x^2 + 25x + 17$	$-4 + i$
59.	$h(x) = x^4 - 6x^3 + 14x^2 - 18x + 9$	$1 - \sqrt{2}i$
60.	$h(x) = x^4 + x^3 - 3x^2 - 13x + 14$	$-2 + \sqrt{3}i$

Finding the Zeros of a Polynomial Function In Exercises 61–72, write the polynomial as the product of linear factors and list all the zeros of the function.

61. $f(x) = x^2 + 36$
62. $f(x) = x^2 + 49$
63. $h(x) = x^2 - 2x + 17$
64. $g(x) = x^2 + 10x + 17$
65. $f(x) = x^4 - 16$
66. $f(y) = y^4 - 256$
67. $f(z) = z^2 - 2z + 2$
68. $h(x) = x^3 - 3x^2 + 4x - 2$
69. $g(x) = x^3 - 3x^2 + x + 5$
70. $f(x) = x^3 - x^2 + x + 39$
71. $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$
72. $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

Finding the Zeros of a Polynomial Function In Exercises 73–78, find all the zeros of the function. When there is an extended list of possible rational zeros, use a graphing utility to graph the function in order to disregard any of the possible rational zeros that are obviously not zeros of the function.

73. $f(x) = x^3 + 24x^2 + 214x + 740$
74. $f(s) = 2s^3 - 5s^2 + 12s - 5$
75. $f(x) = 16x^3 - 20x^2 - 4x + 15$
76. $f(x) = 9x^3 - 15x^2 + 11x - 5$
77. $f(x) = 2x^4 + 5x^3 + 4x^2 + 5x + 2$
78. $g(x) = x^5 - 8x^4 + 28x^3 - 56x^2 + 64x - 32$

Using Descartes's Rule of Signs In Exercises 79–86, use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of the function.

79. $g(x) = 2x^3 - 3x^2 - 3$
80. $h(x) = 4x^2 - 8x + 3$
81. $h(x) = 2x^3 + 3x^2 + 1$
82. $h(x) = 2x^4 - 3x - 2$
83. $g(x) = 6x^4 + 2x^3 - 3x^2 + 2$
84. $f(x) = 4x^3 - 3x^2 - 2x - 1$
85. $f(x) = 5x^3 + x^2 - x + 5$
86. $f(x) = 3x^3 - 2x^2 - x + 3$

Verifying Upper and Lower Bounds In Exercises 87–90, use synthetic division to verify the upper and lower bounds of the real zeros of f.

87. $f(x) = x^3 + 3x^2 - 2x + 1$
 (a) Upper: $x = 1$ (b) Lower: $x = -4$
88. $f(x) = x^3 - 4x^2 + 1$
 (a) Upper: $x = 4$ (b) Lower: $x = -1$
89. $f(x) = x^4 - 4x^3 + 16x - 16$
 (a) Upper: $x = 5$ (b) Lower: $x = -3$
90. $f(x) = 2x^4 - 8x + 3$
 (a) Upper: $x = 3$ (b) Lower: $x = -4$

Finding Real Zeros of a Polynomial Function In Exercises 91–94, find all real zeros of the function.

91. $f(x) = 16x^3 - 12x^2 - 4x + 3$
92. $f(z) = 12z^3 - 4z^2 - 27z + 9$
93. $f(y) = 4y^3 + 3y^2 + 8y + 6$
94. $g(x) = 3x^3 - 2x^2 + 15x - 10$

Finding the Rational Zeros of a Polynomial In Exercises 95–98, find the rational zeros of the polynomial function.

95. $P(x) = x^4 - \frac{25}{4}x^2 + 9 = \frac{1}{4}(4x^4 - 25x^2 + 36)$
96. $f(x) = x^3 - \frac{3}{2}x^2 - \frac{23}{2}x + 6$
 $= \frac{1}{2}(2x^3 - 3x^2 - 23x + 12)$
97. $f(x) = x^3 - \frac{1}{4}x^2 - x + \frac{1}{4} = \frac{1}{4}(4x^3 - x^2 - 4x + 1)$
98. $f(z) = z^3 + \frac{11}{6}z^2 - \frac{1}{2}z - \frac{1}{3} = \frac{1}{6}(6z^3 + 11z^2 - 3z - 2)$

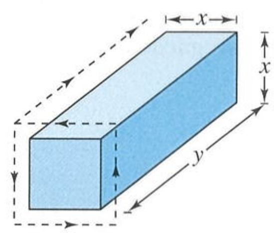
Rational and Irrational Zeros In Exercises 99–102, match the cubic function with the numbers of rational and irrational zeros.

(a) Rational zeros: 0; irrational zeros: 1
(b) Rational zeros: 3; irrational zeros: 0
(c) Rational zeros: 1; irrational zeros: 2
(d) Rational zeros: 1; irrational zeros: 0

99. $f(x) = x^3 - 1$
100. $f(x) = x^3 - 2$
101. $f(x) = x^3 - x$
102. $f(x) = x^3 - 2x$

103. **Geometry** You want to make an open box from a rectangular piece of material, 15 centimeters by 9 centimeters, by cutting equal squares from the corners and turning up the sides.
 (a) Let x represent the side length of each of the squares removed. Draw a diagram showing the squares removed from the original piece of material and the resulting dimensions of the open box.
 (b) Use the diagram to write the volume V of the box as a function of x. Determine the domain of the function.
 (c) Sketch the graph of the function and approximate the dimensions of the box that yield a maximum volume.
 (d) Find values of x such that $V = 56$. Which of these values is a physical impossibility in the construction of the box? Explain.

104. **Geometry** A rectangular package to be sent by a delivery service (see figure) has a combined length and girth (perimeter of a cross section) of 120 inches.



 (a) Use the diagram to write the volume V of the package as a function of x.
 (b) Use a graphing utility to graph the function and approximate the dimensions of the package that yield a maximum volume.
 (c) Find values of x such that $V = 13{,}500$. Which of these values is a physical impossibility in the construction of the package? Explain.

105. **Geometry**
A bulk food storage bin with dimensions 2 feet by 3 feet by 4 feet needs to be increased in size to hold five times as much food as the current bin.
 (a) Assume each dimension is increased by the same amount. Write a function that represents the volume V of the new bin.
 (b) Find the dimensions of the new bin.

106. Cost The ordering and transportation cost C (in thousands of dollars) for machine parts is given by

$$C(x) = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \geq 1$$

where x is the order size (in hundreds). In calculus, it can be shown that the cost is a minimum when

$$3x^3 - 40x^2 - 2400x - 36{,}000 = 0.$$

Use a graphing utility to approximate the optimal order size to the nearest hundred units.

Exploration

True or False? **In Exercises 107 and 108, decide whether the statement is true or false. Justify your answer.**

107. It is possible for a third-degree polynomial function with integer coefficients to have no real zeros.

108. If $x = -i$ is a zero of the function

$$f(x) = x^3 + ix^2 + ix - 1$$

then $x = i$ must also be a zero of f.

Think About It **In Exercises 109–114, determine (if possible) the zeros of the function g when the function f has zeros at $x = r_1$, $x = r_2$, and $x = r_3$.**

109. $g(x) = -f(x)$

110. $g(x) = 3f(x)$

111. $g(x) = f(x - 5)$

112. $g(x) = f(2x)$

113. $g(x) = 3 + f(x)$

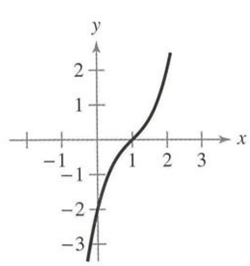
114. $g(x) = f(-x)$

115. Think About It A cubic polynomial function f has real zeros -2, $\frac{1}{2}$, and 3, and its leading coefficient is negative. Write an equation for f and sketch its graph. How many different polynomial functions are possible for f?

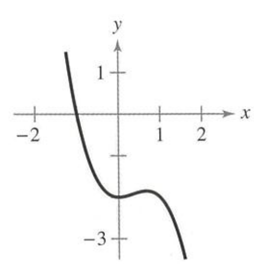
116. Think About It Sketch the graph of a fifth-degree polynomial function whose leading coefficient is positive and that has a zero at $x = 3$ of multiplicity 2.

Writing an Equation **In Exercises 117 and 118, the graph of a cubic polynomial function $y = f(x)$ is shown. One of the zeros is $1 + i$. Write an equation for f.**

117.

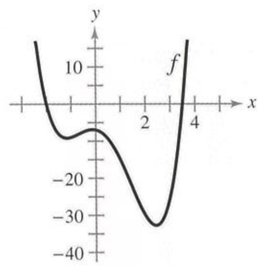


118.



119. Error Analysis Describe the error.

The graph of a quartic (fourth-degree) polynomial $y = f(x)$ is shown. One of the zeros is i.



The function is $f(x) = (x + 2)(x - 3.5)(x - i)$. X

120. HOW DO YOU SEE IT? Use the information in the table to answer each question.

Interval	Value of $f(x)$
$(-\infty, -2)$	Positive
$(-2, 1)$	Negative
$(1, 4)$	Negative
$(4, \infty)$	Positive

(a) What are the three real zeros of the polynomial function f?

(b) What can be said about the behavior of the graph of f at $x = 1$?

(c) What is the least possible degree of f? Explain. Can the degree of f ever be odd? Explain.

(d) Is the leading coefficient of f positive or negative? Explain.

(e) Sketch a graph of a function that exhibits the behavior described in the table.

121. Think About It Let $y = f(x)$ be a quartic (fourth-degree) polynomial with leading coefficient $a = 1$ and

$$f(i) = f(2i) = 0.$$

Write an equation for f.

122. Think About It Let $y = f(x)$ be a cubic polynomial with leading coefficient $a = -1$ and

$$f(2) = f(i) = 0.$$

Write an equation for f.

123. Writing an Equation Write the equation for a quadratic function f (with integer coefficients) that has the given zeros. Assume that b is a positive integer.

(a) $\pm\sqrt{b}i$ (b) $a \pm bi$

2.6 Rational Functions

Rational functions have many real-life applications. For example, in Exercise 69 on page 176, you will use a rational function to determine the cost of supplying recycling bins to the population of a rural township.

- **Find domains of rational functions.**
- **Find vertical and horizontal asymptotes of graphs of rational functions.**
- **Sketch graphs of rational functions.**
- **Sketch graphs of rational functions that have slant asymptotes.**
- **Use rational functions to model and solve real-life problems.**

Introduction

A **rational function** is a quotient of polynomial functions. It can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

The *domain* of a rational function of x includes all real numbers except x-values that make the denominator zero. Much of the discussion of rational functions will focus on the behavior of their graphs near x-values excluded from the domain.

EXAMPLE 1 Finding the Domain of a Rational Function

See LarsonPrecalculus.com for an interactive version of this type of example.

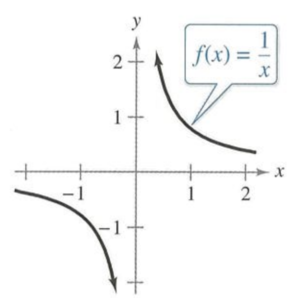
Find the domain of $f(x) = \frac{1}{x}$ and discuss the behavior of f near any excluded x-values.

Solution The denominator is zero when $x = 0$, so the domain of f is all real numbers except $x = 0$. To determine the behavior of f near this excluded value, evaluate $f(x)$ to the left and right of $x = 0$, as shown in the tables below.

x	-1	-0.5	-0.1	-0.01	-0.001	$\rightarrow 0$
$f(x)$	-1	-2	-10	-100	-1000	$\rightarrow -\infty$

x	$0 \leftarrow$	0.001	0.01	0.1	0.5	1
$f(x)$	$\infty \leftarrow$	1000	100	10	2	1

Note that as x approaches 0 *from the left,* $f(x)$ decreases without bound. In contrast, as x approaches 0 *from the right,* $f(x)$ increases without bound. The graph of f is shown below.



• • **REMARK** Recall from Section 1.6 that the rational function

$$f(x) = \frac{1}{x}$$

is also referred to as the *reciprocal function.*

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Find the domain of $f(x) = \frac{3x}{x - 1}$ and discuss the behavior of f near any excluded x-values.

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Vertical and Horizontal Asymptotes

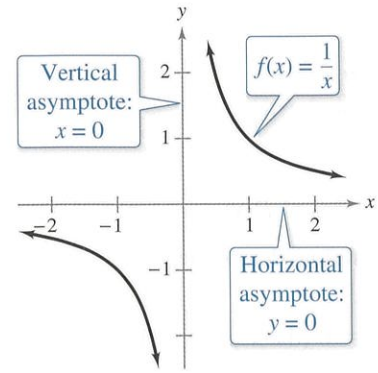


Figure 2.19

In Example 1, the behavior of f near $x = 0$ is as denoted below.

$f(x) \to -\infty$ as $x \to 0^-$ — $f(x)$ decreases without bound as x approaches 0 from the left.

$f(x) \to \infty$ as $x \to 0^+$ — $f(x)$ increases without bound as x approaches 0 from the right.

The line $x = 0$ is a **vertical asymptote** of the graph of f, as shown in Figure 2.19. Notice that the graph of f also has a **horizontal asymptote**—the line $y = 0$. The behavior of f near $y = 0$ is as denoted below.

$f(x) \to 0$ as $x \to -\infty$ — $f(x)$ approaches 0 as x decreases without bound.

$f(x) \to 0$ as $x \to \infty$ — $f(x)$ approaches 0 as x increases without bound.

Definitions of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f when

$$f(x) \to \infty \quad \text{or} \quad f(x) \to -\infty$$

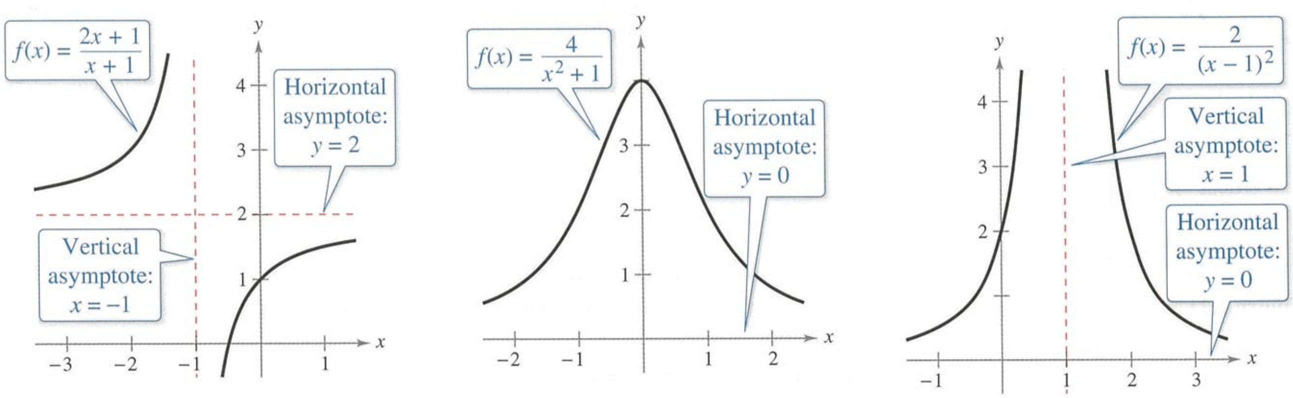
as $x \to a$, either from the right or from the left.

2. The line $y = b$ is a **horizontal asymptote** of the graph of f when

$$f(x) \to b$$

as $x \to \infty$ or $x \to -\infty$.

Eventually (as $x \to \infty$ or $x \to -\infty$), the distance between the horizontal asymptote and the points on the graph must approach zero. Figure 2.20 shows the vertical and horizontal asymptotes of the graphs of three rational functions.



(a) (b) (c)

Figure 2.20

Verify numerically the horizontal asymptotes shown in Figure 2.20. For example, to show that the line $y = 2$ is the horizontal asymptote of the graph of

$$f(x) = \frac{2x + 1}{x + 1}$$

create a table that shows the value of $f(x)$ as x increases and decreases without bound.

The graphs of $f(x) = \dfrac{1}{x}$ in Figure 2.19 and $f(x) = \dfrac{2x + 1}{x + 1}$ in Figure 2.20(a) are **hyperbolas.** You will study hyperbolas in Chapter 10.

Vertical and Horizontal Asymptotes

Let f be the rational function

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of $D(x)$.

2. The graph of f has at most one *horizontal* asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.

a. When $n < m$, the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.

b. When $n = m$, the graph of f has the line $y = \frac{a_n}{b_m}$ (ratio of the leading coefficients) as a horizontal asymptote.

c. When $n > m$, the graph of f has no horizontal asymptote.

EXAMPLE 2 Finding Vertical and Horizontal Asymptotes

Find all vertical and horizontal asymptotes of the graph of each rational function.

a. $f(x) = \dfrac{2x^2}{x^2 - 1}$ **b.** $f(x) = \dfrac{x^2 + x - 2}{x^2 - x - 6}$

Solution

a. For this rational function, the degree of the numerator is *equal* to the degree of the denominator. The leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 1, so the graph has the line $y = 2/1 = 2$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$x^2 - 1 = 0$ Set denominator equal to zero.

$(x + 1)(x - 1) = 0$ Factor.

$x + 1 = 0 \Rightarrow x = -1$ Set 1st factor equal to 0.

$x - 1 = 0 \Rightarrow x = 1$ Set 2nd factor equal to 0.

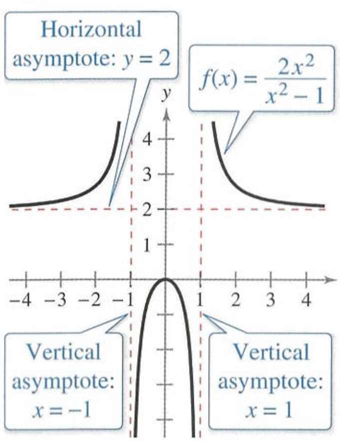


Figure 2.21

This equation has two real solutions, $x = -1$ and $x = 1$, so the graph has the lines $x = -1$ and $x = 1$ as vertical asymptotes. Figure 2.21 shows the graph of this function.

b. For this rational function, the degree of the numerator is equal to the degree of the denominator. The leading coefficients of the numerator and the denominator are both 1, so the graph has the line $y = 1/1 = 1$ as a horizontal asymptote. To find any vertical asymptotes, first factor the numerator and denominator as follows.

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6} = \frac{(x - 1)\cancel{(x + 2)}}{\cancel{(x + 2)}(x - 3)} = \frac{x - 1}{x - 3}, \quad x \neq -2$$

Setting the denominator $x - 3$ (of the simplified function) equal to zero, you find that the graph has the line $x = 3$ as a vertical asymptote.

REMARK There is a *hole* in the graph of f at $x = -2$. In Example 6, you will sketch the graph of a rational function that has a hole.

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Find all vertical and horizontal asymptotes of the graph of $f(x) = \dfrac{3x^2 + 7x - 6}{x^2 + 4x + 3}$.

Sketching the Graph of a Rational Function

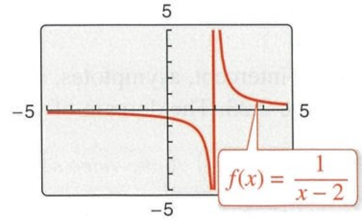
To sketch the graph of a rational function, use the following guidelines.

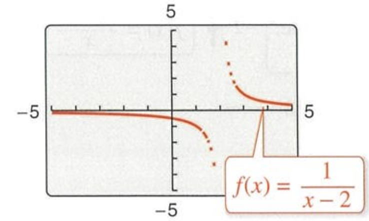
> **Guidelines for Graphing Rational Functions**
>
> Let $f(x) = \dfrac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.
>
> 1. Simplify f, if possible. List any restrictions on the domain of f that are not implied by the simplified function.
> 2. Find and plot the y-intercept (if any) by evaluating $f(0)$.
> 3. Find the zeros of the numerator (if any). Then plot the corresponding x-intercepts.
> 4. Find the zeros of the denominator (if any). Then sketch the corresponding vertical asymptotes.
> 5. Find and sketch the horizontal asymptote (if any) by using the rule for finding the horizontal asymptote of a rational function on page 168.
> 6. Plot at least one point *between* and one point *beyond* each x-intercept and vertical asymptote.
> 7. Use smooth curves to complete the graph between and beyond the vertical asymptotes.

The concept of *test intervals* from Section 2.2 can be extended to graphing rational functions. Be aware, however, that although a polynomial function can change signs only at its zeros, a rational function can change signs both at its zeros and at its undefined values (the x-values for which its denominator is zero). So, to form the test intervals in which a rational function has no sign changes, arrange the x-values representing the zeros of both the numerator and the denominator of the rational function in increasing order.

You may also want to test for symmetry when graphing rational functions, especially for simple rational functions. Recall from Section 1.6 that the graph of the reciprocal function $f(x) = \dfrac{1}{x}$ is symmetric with respect to the origin.

TECHNOLOGY Some graphing utilities have difficulty graphing rational functions with vertical asymptotes. In connected mode, the graphing utility may connect portions of the graph that are not supposed to be connected. For example, the graph on the left should consist of two unconnected portions—one to the left of $x = 2$ and the other to the right of $x = 2$. Changing the mode of the graphing utility to *dot* mode eliminates this problem. In *dot* mode, however, the graph is represented as a collection of dots (as shown in the graph on the right) rather than as a smooth curve.





EXAMPLE 3 Sketching the Graph of a Rational Function

Sketch the graph of $g(x) = \dfrac{3}{x-2}$ and state its domain.

Solution

y-intercept: $\left(0, -\frac{3}{2}\right)$, because $g(0) = -\frac{3}{2}$

x-intercept: none, because there are no zeros of the numerator

Vertical asymptote: $x = 2$, zero of denominator

Horizontal asymptote: $y = 0$, because degree of $N(x) <$ degree of $D(x)$

Additional points:

Test Interval	Representative x-Value	Value of g	Sign	Point on Graph
$(-\infty, 2)$	-4	$g(-4) = -\frac{1}{2}$	Negative	$\left(-4, -\frac{1}{2}\right)$
$(2, \infty)$	3	$g(3) = 3$	Positive	$(3, 3)$

REMARK You can use transformations to help you sketch graphs of rational functions. For instance, the graph of g in Example 3 is a vertical stretch and a right shift of the graph of $f(x) = 1/x$ because

$$g(x) = \frac{3}{x-2} = 3\left(\frac{1}{x-2}\right) = 3f(x-2).$$

By plotting the intercept, asymptotes, and a few additional points, you obtain the graph shown in Figure 2.22. The domain of g is all real numbers except $x = 2$.

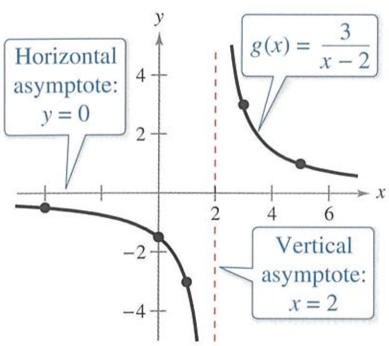


Figure 2.22

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Sketch the graph of $f(x) = \dfrac{1}{x+3}$ and state its domain.

EXAMPLE 4 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = (2x - 1)/x$ and state its domain.

Solution

y-intercept: none, because $x = 0$ is not in the domain

x-intercept: $\left(\frac{1}{2}, 0\right)$, because $2x - 1 = 0$ when $x = \frac{1}{2}$

Vertical asymptote: $x = 0$, zero of denominator

Horizontal asymptote: $y = 2$, because degree of $N(x) =$ degree of $D(x)$

Additional points:

Test Interval	Representative x-Value	Value of f	Sign	Point on Graph
$(-\infty, 0)$	-1	$f(-1) = 3$	Positive	$(-1, 3)$
$\left(0, \frac{1}{2}\right)$	$\frac{1}{4}$	$f\left(\frac{1}{4}\right) = -2$	Negative	$\left(\frac{1}{4}, -2\right)$
$\left(\frac{1}{2}, \infty\right)$	4	$f(4) = \frac{7}{4}$	Positive	$\left(4, \frac{7}{4}\right)$

By plotting the intercept, asymptotes, and a few additional points, you obtain the graph shown in Figure 2.23. The domain of f is all real numbers except $x = 0$.

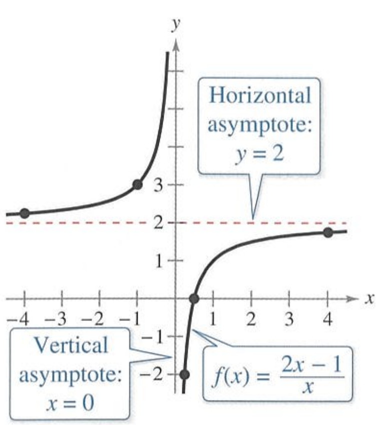

Figure 2.23

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Sketch the graph of $g(x) = (3 + 2x)/(1 + x)$ and state its domain.

EXAMPLE 5 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = x/(x^2 - x - 2)$.

Solution Factoring the denominator, you have $f(x) = x/[(x + 1)(x - 2)]$.

Intercept: $(0, 0)$, because $f(0) = 0$

Vertical asymptotes: $x = -1$, $x = 2$, zeros of denominator

Horizontal asymptote: $y = 0$, because degree of $N(x) <$ degree of $D(x)$

Additional points:

Test Interval	Representative x-Value	Value of f	Sign	Point on Graph
$(-\infty, -1)$	-3	$f(-3) = \frac{3}{10}$	Negative	$\left(-3, -\frac{3}{10}\right)$
$(-1, 0)$	$-\frac{1}{2}$	$f\left(-\frac{1}{2}\right) = \frac{2}{5}$	Positive	$\left(-\frac{1}{2}, \frac{2}{5}\right)$
$(0, 2)$	1	$f(1) = -\frac{1}{2}$	Negative	$\left(1, -\frac{1}{2}\right)$
$(2, \infty)$	3	$f(3) = \frac{3}{4}$	Positive	$\left(3, \frac{3}{4}\right)$

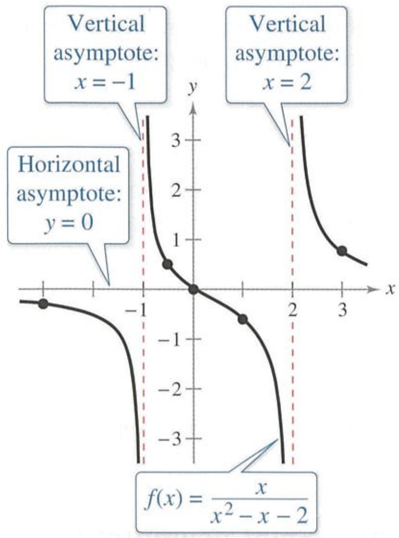

Figure 2.24

Figure 2.24 shows the graph of this function.

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Sketch the graph of $f(x) = 3x/(x^2 + x - 2)$.

REMARK If you are unsure of the shape of a portion of the graph of a rational function, then plot some additional points. Also note that when the numerator and the denominator of a rational function have a common factor, the graph of the function has a *hole* at the zero of the common factor. (See Example 6.)

EXAMPLE 6 A Rational Function with Common Factors

Sketch the graph of $f(x) = (x^2 - 9)/(x^2 - 2x - 3)$.

Solution By factoring the numerator and denominator, you have

$$f(x) = \frac{x^2 - 9}{x^2 - 2x - 3} = \frac{\cancel{(x - 3)}(x + 3)}{\cancel{(x - 3)}(x + 1)} = \frac{x + 3}{x + 1}, \quad x \neq 3.$$

y-intercept: $(0, 3)$, because $f(0) = 3$

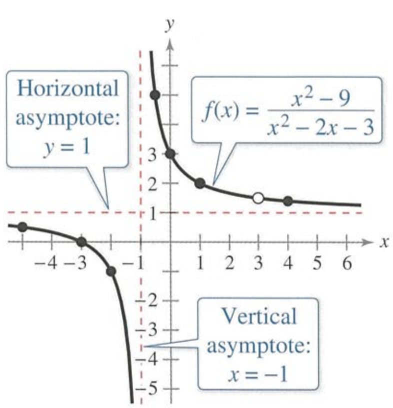
x-intercept: $(-3, 0)$, because $x + 3 = 0$ when $x = -3$

Vertical asymptote: $x = -1$, zero of (simplified) denominator

Horizontal asymptote: $y = 1$, because degree of $N(x) =$ degree of $D(x)$

Additional points:

Test Interval	Representative x-Value	Value of f	Sign	Point on Graph
$(-\infty, -3)$	-4	$f(-4) = \frac{1}{3}$	Positive	$\left(-4, \frac{1}{3}\right)$
$(-3, -1)$	-2	$f(-2) = -1$	Negative	$(-2, -1)$
$(-1, \infty)$	2	$f(2) = \frac{5}{3}$	Positive	$\left(2, \frac{5}{3}\right)$



Hole at $x = 3$

Figure 2.25

Figure 2.25 shows the graph of this function. Notice that there is a hole in the graph at $x = 3$, because the numerator and denominator have a common factor of $x - 3$.

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Sketch the graph of $f(x) = (x^2 - 4)/(x^2 - x - 6)$.

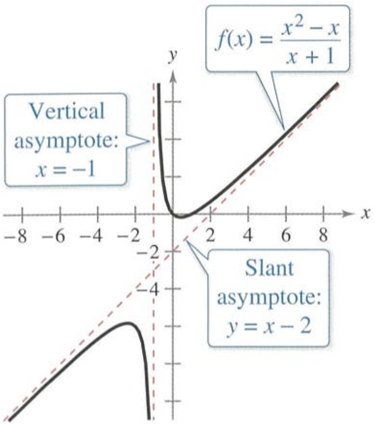


Figure 2.26

Slant Asymptotes

Consider a rational function whose denominator is of degree 1 or greater. If the degree of the numerator is exactly *one more* than the degree of the denominator, then the graph of the function has a **slant** (or **oblique**) **asymptote.** For example, the graph of

$$f(x) = \frac{x^2 - x}{x + 1}$$

has a slant asymptote, as shown in Figure 2.26. To find the equation of a slant asymptote, use long division. For example, by dividing $x + 1$ into $x^2 - x$, you obtain

$$f(x) = \frac{x^2 - x}{x + 1} = \underbrace{x - 2}_{\substack{\text{Slant asymptote} \\ (y = x - 2)}} + \frac{2}{x + 1}.$$

As x increases or decreases without bound, the remainder term $2/(x + 1)$ approaches 0, so the graph of f approaches the line $y = x - 2$, as shown in Figure 2.26.

EXAMPLE 7 A Rational Function with a Slant Asymptote

Sketch the graph of $f(x) = \dfrac{x^2 - x - 2}{x - 1}$.

Solution Factoring the numerator as $(x - 2)(x + 1)$ enables you to recognize the x-intercepts. Using long division

$$f(x) = \frac{x^2 - x - 2}{x - 1} = x - \frac{2}{x - 1}$$

enables you to recognize that the line $y = x$ is a slant asymptote of the graph.

y-intercept: $(0, 2)$, because $f(0) = 2$

x-intercepts: $(2, 0)$ and $(-1, 0)$, because $x - 2 = 0$ when $x = 2$ and $x + 1 = 0$ when $x = -1$

Vertical asymptote: $x = 1$, zero of denominator

Slant asymptote: $y = x$

Additional points:

Test Interval	Representative x-Value	Value of f	Sign	Point on Graph
$(-\infty, -1)$	-2	$f(-2) = -\frac{4}{3}$	Negative	$\left(-2, -\frac{4}{3}\right)$
$(-1, 1)$	$\frac{1}{2}$	$f\left(\frac{1}{2}\right) = \frac{9}{2}$	Positive	$\left(\frac{1}{2}, \frac{9}{2}\right)$
$(1, 2)$	$\frac{3}{2}$	$f\left(\frac{3}{2}\right) = -\frac{5}{2}$	Negative	$\left(\frac{3}{2}, -\frac{5}{2}\right)$
$(2, \infty)$	3	$f(3) = 2$	Positive	$(3, 2)$

Figure 2.27 shows the graph of the function.

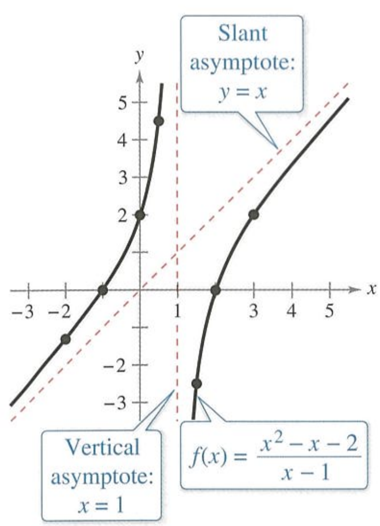


Figure 2.27

✔ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Sketch the graph of $f(x) = \dfrac{3x^2 + 1}{x}$.

Application

There are many examples of asymptotic behavior in real life. For instance, Example 8 shows how a vertical asymptote can help you to analyze the cost of removing pollutants from smokestack emissions.

EXAMPLE 8 Cost-Benefit Model

A utility company burns coal to generate electricity. The cost C (in dollars) of removing $p\%$ of the smokestack pollutants is given by

$$C = \frac{80{,}000p}{100 - p}, \quad 0 \le p \le 100.$$

You are a member of a state legislature considering a law that would require utility companies to remove 90% of the pollutants from their smokestack emissions. The current law requires 85% removal. How much additional cost would the utility company incur as a result of the new law?

Algebraic Solution

The current law requires 85% removal, so the current cost to the utility company is

$$C = \frac{80{,}000(85)}{100 - 85} \qquad \text{Evaluate } C \text{ when } p = 85.$$

$$\approx \$453{,}333.$$

The cost to remove 90% of the pollutants would be

$$C = \frac{80{,}000(90)}{100 - 90} \qquad \text{Evaluate } C \text{ when } p = 90.$$

$$= \$720{,}000.$$

So, the new law would require the utility company to spend an additional

$$720{,}000 - 453{,}333 = \$266{,}667. \qquad \text{Subtract 85\% removal cost from 90\% removal cost.}$$

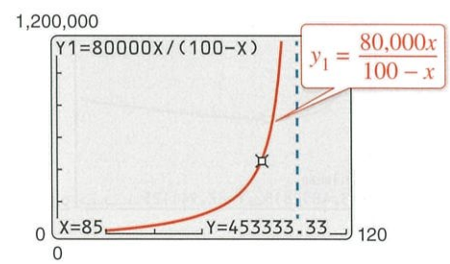
Graphical Solution

Use a graphing utility to graph the function

$$y_1 = \frac{80{,}000x}{100 - x}$$

and use the *value* feature to approximate the values of y_1 when $x = 85$ and $x = 90$, as shown below. Note that the graph has a vertical asymptote at

$$x = 100.$$



When $x = 85$, $y_1 \approx 453{,}333$.

When $x = 90$, $y_1 = 720{,}000$.

So, the new law would require the utility company to spend an additional

$$720{,}000 - 453{,}333 = \$266{,}667.$$

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The cost C (in millions of dollars) of removing $p\%$ of the industrial and municipal pollutants discharged into a river is given by

$$C = \frac{255p}{100 - p}, \quad 0 \le p < 100.$$

a. Find the costs of removing 20%, 45%, and 80% of the pollutants.

b. According to the model, is it possible to remove 100% of the pollutants? Explain.

EXAMPLE 9 Finding a Minimum Area

A rectangular page contains 48 square inches of print. The margins at the top and bottom of the page are each 1 inch deep. The margins on each side are $1\frac{1}{2}$ inches wide. What should the dimensions of the page be to use the least amount of paper?

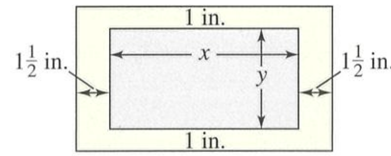


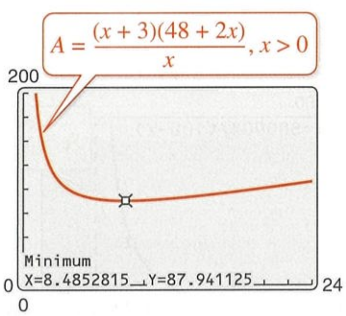
Figure 2.28

Graphical Solution

Let A be the area to be minimized. From Figure 2.28, you can write $A = (x + 3)(y + 2)$. The printed area inside the margins is given by $xy = 48$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

The graph of this rational function is shown below. Because x represents the width of the printed area, you need to consider only the portion of the graph for which x is positive. Use the *minimum* feature of a graphing utility to estimate that the minimum value of A occurs when $x \approx 8.5$ inches. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be $x + 3 \approx 11.5$ inches by $y + 2 \approx 7.6$ inches.



Numerical Solution

Let A be the area to be minimized. From Figure 2.28, you can write $A = (x + 3)(y + 2)$. The printed area inside the margins is given by $xy = 48$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

Use the *table* feature of a graphing utility to create a table of values for the function $y_1 = [(x + 3)(48 + 2x)]/x$ beginning at $x = 1$ and increasing by 1. The minimum value of y_1 occurs when x is somewhere between 8 and 9, as shown in Figure 2.29. To approximate the minimum value of y_1 to one decimal place, change the table to begin at $x = 8$ and increase by 0.1. The minimum value of y_1 occurs when $x \approx 8.5$, as shown in Figure 2.30. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be $x + 3 \approx 11.5$ inches by $y + 2 \approx 7.6$ inches.

X	Y1	
6	90	
7	88.571	
8	88	
9	88	
10	88.4	
11	89.091	
12	90	
X=8		

Figure 2.29

X	Y1	
8.2	87.961	
8.3	87.949	
8.4	87.943	
8.5	87.941	
8.6	87.944	
8.7	87.952	
8.8	87.964	
X=8.5		

Figure 2.30

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Rework Example 9 when the margins on each side are 2 inches wide and the page contains 40 square inches of print.

Summarize (Section 2.6)

1. State the definition of a rational function and describe the domain *(page 166)*. For an example of finding the domain of a rational function, see Example 1.
2. Explain how to find the vertical and horizontal asymptotes of the graph of a rational function *(page 168)*. For an example of finding vertical and horizontal asymptotes of graphs of rational functions, see Example 2.
3. Explain how to sketch the graph of a rational function *(page 169)*. For examples of sketching the graphs of rational functions, see Examples 3–6.
4. Explain how to determine whether the graph of a rational function has a slant asymptote *(page 172)*. For an example of sketching the graph of a rational function that has a slant asymptote, see Example 7.
5. Describe examples of how to use rational functions to model and solve real-life problems *(pages 173 and 174, Examples 8 and 9)*.

2.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. Functions of the form $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial, are called ________ ________.
2. When $f(x) \to \pm\infty$ as $x \to a$ from the left or the right, $x = a$ is a ________ ________ of the graph of f.
3. When $f(x) \to b$ as $x \to \pm\infty$, $y = b$ is a ________ ________ of the graph of f.
4. For the rational function $f(x) = N(x)/D(x)$, if the degree of $N(x)$ is exactly one more than the degree of $D(x)$, then the graph of f has a ________ (or oblique) ________.

Skills and Applications

Finding the Domain of a Rational Function In Exercises 5–8, find the domain of the function and discuss the behavior of f near any excluded x-values.

5. $f(x) = \dfrac{1}{x-1}$
6. $f(x) = \dfrac{5x}{x+2}$
7. $f(x) = \dfrac{3x^2}{x^2-1}$
8. $f(x) = \dfrac{2x}{x^2-4}$

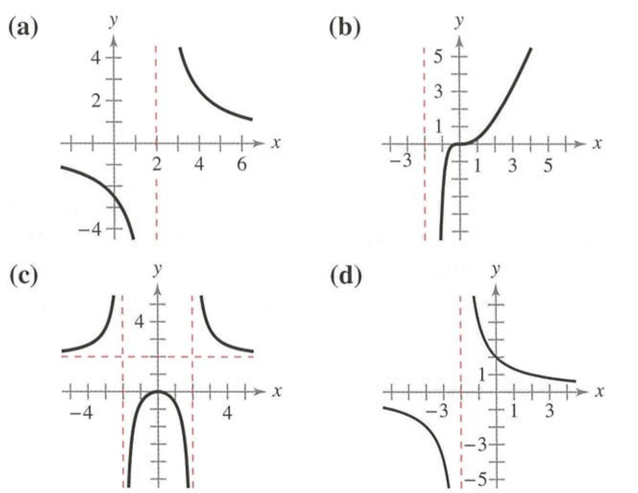
Finding Vertical and Horizontal Asymptotes In Exercises 9–16, find all vertical and horizontal asymptotes of the graph of the function.

9. $f(x) = \dfrac{4}{x^2}$
10. $f(x) = \dfrac{1}{(x-2)^3}$
11. $f(x) = \dfrac{5+x}{5-x}$
12. $f(x) = \dfrac{3-7x}{3+2x}$
13. $f(x) = \dfrac{x^3}{x^2-x}$
14. $f(x) = \dfrac{4x^2}{x+2}$
15. $f(x) = \dfrac{x^2-3x-4}{2x^2+x-1}$
16. $f(x) = \dfrac{-4x^2+1}{x^2+x+3}$

Sketching the Graph of a Rational Function In Exercises 17–38, (a) state the domain of the function, (b) identify all intercepts, (c) find any vertical or horizontal asymptotes, and (d) plot additional solution points as needed to sketch the graph of the rational function.

17. $f(x) = \dfrac{1}{x+1}$
18. $f(x) = \dfrac{1}{x-3}$
19. $h(x) = \dfrac{-1}{x+4}$
20. $g(x) = \dfrac{1}{6-x}$
21. $C(x) = \dfrac{2x+3}{x+2}$
22. $P(x) = \dfrac{1-3x}{1-x}$
23. $f(x) = \dfrac{x^2}{x^2+9}$
24. $f(t) = \dfrac{1-2t}{t}$
25. $g(s) = \dfrac{4s}{s^2+4}$
26. $f(x) = -\dfrac{x}{(x-2)^2}$
27. $h(x) = \dfrac{2x}{x^2-3x-4}$
28. $g(x) = \dfrac{3x}{x^2+2x-3}$
29. $f(x) = \dfrac{x-4}{x^2-16}$
30. $f(x) = \dfrac{x+1}{x^2-1}$
31. $f(t) = \dfrac{t^2-1}{t-1}$
32. $f(x) = \dfrac{x^2-36}{x+6}$
33. $f(x) = \dfrac{x^2-25}{x^2-4x-5}$
34. $f(x) = \dfrac{x^2-4}{x^2-3x+2}$
35. $f(x) = \dfrac{x^2+3x}{x^2+x-6}$
36. $f(x) = \dfrac{5(x+4)}{x^2+x-12}$
37. $f(x) = \dfrac{2x^2-5x-3}{x^3-2x^2-x+2}$
38. $f(x) = \dfrac{x^2-x-2}{x^3-2x^2-5x+6}$

Matching In Exercises 39–42, match the rational function with its graph. [The graphs are labeled (a)–(d).]



39. $f(x) = \dfrac{4}{x+2}$
40. $f(x) = \dfrac{5}{x-2}$
41. $f(x) = \dfrac{2x^2}{x^2-4}$
42. $f(x) = \dfrac{3x^3}{(x+2)^2}$

Comparing Graphs of Functions **In Exercises 43–46, (a) state the domains of f and g, (b) use a graphing utility to graph f and g in the same viewing window, and (c) explain why the graphing utility may not show the difference in the domains of f and g.**

43. $f(x) = \dfrac{x^2 - 1}{x + 1}, \quad g(x) = x - 1$

44. $f(x) = \dfrac{x^2(x - 2)}{x^2 - 2x}, \quad g(x) = x$

45. $f(x) = \dfrac{x - 2}{x^2 - 2x}, \quad g(x) = \dfrac{1}{x}$

46. $f(x) = \dfrac{2x - 6}{x^2 - 7x + 12}, \quad g(x) = \dfrac{2}{x - 4}$

A Rational Function with a Slant Asymptote **In Exercises 47–60, (a) state the domain of the function, (b) identify all intercepts, (c) find any vertical or slant asymptotes, and (d) plot additional solution points as needed to sketch the graph of the rational function.**

47. $h(x) = \dfrac{x^2 - 4}{x}$

48. $g(x) = \dfrac{x^2 + 5}{x}$

49. $f(x) = \dfrac{2x^2 + 1}{x}$

50. $f(x) = \dfrac{-x^2 - 2}{x}$

51. $g(x) = \dfrac{x^2 + 1}{x}$

52. $h(x) = \dfrac{x^2}{x - 1}$

53. $f(t) = -\dfrac{t^2 + 1}{t + 5}$

54. $f(x) = \dfrac{x^2 + 1}{x + 1}$

55. $f(x) = \dfrac{x^3}{x^2 - 4}$

56. $g(x) = \dfrac{x^3}{2x^2 - 8}$

57. $f(x) = \dfrac{x^2 - x + 1}{x - 1}$

58. $f(x) = \dfrac{2x^2 - 5x + 5}{x - 2}$

59. $f(x) = \dfrac{2x^3 - x^2 - 2x + 1}{x^2 + 3x + 2}$

60. $f(x) = \dfrac{2x^3 + x^2 - 8x - 4}{x^2 - 3x + 2}$

Using Technology **In Exercises 61–64, use a graphing utility to graph the rational function. State the domain of the function and find any asymptotes. Then zoom out sufficiently far so that the graph appears as a line. Identify the line.**

61. $f(x) = \dfrac{x^2 + 2x - 8}{x + 2}$

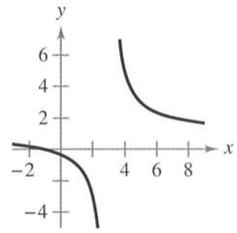
62. $f(x) = \dfrac{2x^2 + x}{x + 1}$

63. $g(x) = \dfrac{1 + 3x^2 - x^3}{x^2}$

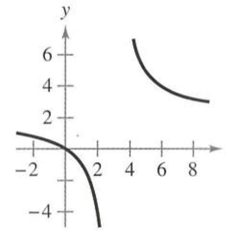
64. $h(x) = \dfrac{12 - 2x - x^2}{2(4 + x)}$

Graphical Reasoning **In Exercises 65–68, (a) use the graph to determine any x-intercepts of the graph of the rational function and (b) set $y = 0$ and solve the resulting equation to confirm your result in part (a).**

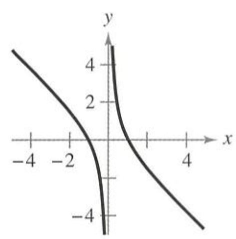
65. $y = \dfrac{x + 1}{x - 3}$



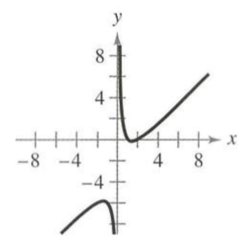
66. $y = \dfrac{2x}{x - 3}$



67. $y = \dfrac{1}{x} - x$



68. $y = x - 3 + \dfrac{2}{x}$



69. **Recycling**

The cost C (in dollars) of supplying recycling bins to $p\%$ of the population of a rural township is given by

$$C = \frac{25{,}000p}{100 - p}, \quad 0 \le p < 100.$$

(a) Use a graphing utility to graph the cost function.

(b) Find the costs of supplying bins to 15%, 50%, and 90% of the population.

(c) According to the model, is it possible to supply bins to 100% of the population? Explain.

70. **Population Growth** The game commission introduces 100 deer into newly acquired state game lands. The population N of the herd is modeled by

$$N = \frac{20(5 + 3t)}{1 + 0.04t}, \quad t \ge 0$$

where t is the time in years.

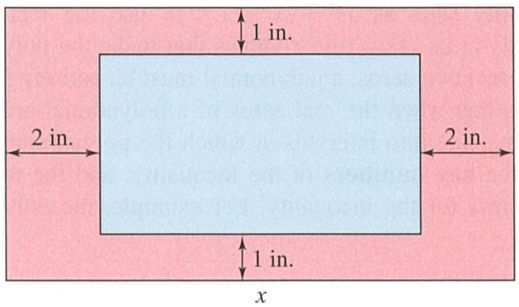
(a) Use a graphing utility to graph this model.

(b) Find the populations when $t = 5$, $t = 10$, and $t = 25$.

(c) What is the limiting size of the herd as time increases?

71. Page Design A rectangular page contains 64 square inches of print. The margins at the top and bottom of the page are each 1 inch deep. The margins on each side are $1\frac{1}{2}$ inches wide. What should the dimensions of the page be to use the least amount of paper?

72. Page Design A page that is x inches wide and y inches high contains 30 square inches of print. The top and bottom margins are each 1 inch deep, and the margins on each side are 2 inches wide (see figure).



(a) Write a functions for the total area A of the page in terms of x.

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Use a graphing utility to graph the area function and approximate the dimensions of the page that use the least amount of paper.

73. Average Speed A driver's average speed is 50 miles per hour on a round trip between two cities 100 miles apart. The average speeds for going and returning were x and y miles per hour, respectively.

(a) Show that $y = (25x)/(x - 25)$.

(b) Determine the vertical and horizontal asymptotes of the graph of the function.

(c) Use a graphing utility to graph the function.

(d) Complete the table.

x	30	35	40	45	50	55	60
y							

(e) Are the results in the table what you expected? Explain.

(f) Is it possible to average 20 miles per hour in one direction and still average 50 miles per hour on the round trip? Explain.

74. Medicine The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is given by

$$C = \frac{3t^2 + t}{t^3 + 50}, \quad t > 0.$$

Use a graphing utility to graph the function. Determine the horizontal asymptote of the graph of the function and interpret its meaning in the context of the problem.

Exploration

True or False? **In Exercises 75–77, determine whether the statement is true or false. Justify your answer.**

75. The graph of a polynomial function can have infinitely many vertical asymptotes.

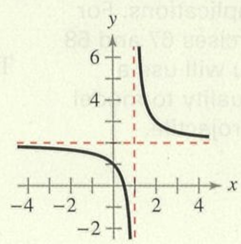
76. The graph of a rational function can never cross one of its asymptotes.

77. The graph of a rational function can have a vertical asymptote, a horizontal asymptote, and a slant asymptote.

78. HOW DO YOU SEE IT? The graph of a rational function

$$f(x) = \frac{N(x)}{D(x)}$$

is shown below. Determine which of the statements about the function is false. Justify your answer.



(a) $D(1) = 0$.

(b) The degree of $N(x)$ and $D(x)$ are equal.

(c) The ratio of the leading coefficients of $N(x)$ and $D(x)$ is 1.

79. Writing Is every rational function a polynomial function? Is every polynomial function a rational function? Explain.

Writing a Rational Function **In Exercises 80–82, write a rational function f whose graph has the specified characteristics. (There are many correct answers.)**

80. Vertical asymptote: None

Horizontal asymptote: $y = 2$

81. Vertical asymptotes: $x = -2, x = 1$

Horizontal asymptote: None

82. Vertical asymptote: $x = 2$

Slant asymptote: $y = x + 1$

Zero of the function: $x = -2$

Project: Department of Defense To work an extended application analyzing the total numbers of military personnel on active duty from 1984 through 2014, visit this text's website at *LarsonPrecalculus.com*. *(Source: U.S. Department of Defense)*

2.7 Nonlinear Inequalities

Nonlinear inequalities have many real-life applications. For example, in Exercises 67 and 68 on page 186, you will use a polynomial inequality to model the height of a projectile.

- **Solve polynomial inequalities.**
- **Solve rational inequalities.**
- **Use nonlinear inequalities to model and solve real-life problems.**

Polynomial Inequalities

To solve a polynomial inequality such as $x^2 - 2x - 3 < 0$, use the fact that a polynomial can change signs only at its *zeros* (the x-values that make the polynomial equal to zero). Between two consecutive zeros, a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into intervals in which the polynomial has no sign changes. These zeros are the **key numbers** of the inequality, and the resulting open intervals are the *test intervals* for the inequality. For example, the polynomial $x^2 - 2x - 3$ factors as

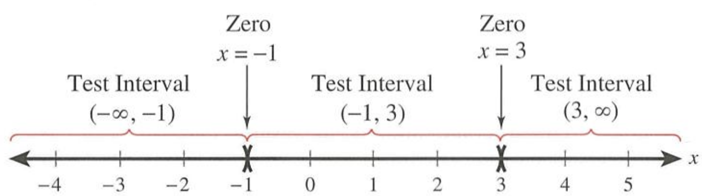
$$x^2 - 2x - 3 = (x + 1)(x - 3)$$

so it has two zeros,

$$x = -1 \quad \text{and} \quad x = 3.$$

These zeros divide the real number line into three test intervals:

$$(-\infty, -1), \quad (-1, 3), \quad \text{and} \quad (3, \infty). \qquad \text{(See figure below.)}$$



Three test intervals for $x^2 - 2x - 3$

To solve the inequality $x^2 - 2x - 3 < 0$, you need to test only one value from each of these test intervals. When a value from a test interval satisfies the original inequality, you can conclude that the interval is a solution of the inequality.

Use the same basic approach, generalized below, to find the solution set of any polynomial inequality.

REMARK The solution set of

$$x^2 - 2x - 3 < 0$$

discussed above, is the open interval $(-1, 3)$. Use Step 3 to verify this. By choosing the representative x-values $x = -2$, $x = 0$, and $x = 4$, you will find that the value of the polynomial is negative only in $(-1, 3)$.

Test Intervals for a Polynomial Inequality

To determine the intervals on which the values of a polynomial are entirely negative or entirely positive, use the steps below.

1. Find all real zeros of the polynomial, and arrange the zeros in increasing order. These zeros are the key numbers of the inequality.
2. Use the key numbers of the inequality to determine the test intervals.
3. Choose one representative x-value in each test interval and evaluate the polynomial at that value. When the value of the polynomial is negative, the polynomial has negative values for every x-value in the interval. When the value of the polynomial is positive, the polynomial has positive values for every x-value in the interval.