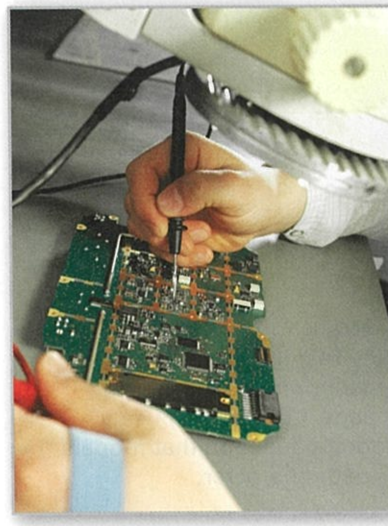
2 Polynomial and Rational Functions

2.1 Quadratic Functions and Models
2.2 Polynomial Functions of Higher Degree
2.3 Polynomial and Synthetic Division
2.4 Complex Numbers
2.5 Zeros of Polynomial Functions
2.6 Rational Functions
2.7 Nonlinear Inequalities

Candle Making Kits (Example 12, page 161)



Electrical Circuit
(Example 87, page 151)

Lyme Disease (Exercise 82, page 144)

Tree Growth
(Exercise 98, page 135)

Path of a Diver (Exercise 67, page 121)

2.1 Quadratic Functions and Models

Quadratic functions have many real-life applications. For example, in Exercise 67 on page 121, you will use a quadratic function that models the path of a diver.

- **Analyze graphs of quadratic functions.**
- **Write quadratic functions in standard form and use the results to sketch their graphs.**
- **Find minimum and maximum values of quadratic functions in real-life applications.**

The Graph of a Quadratic Function

In this and the next section, you will study graphs of polynomial functions. Section 1.6 introduced basic functions such as linear, constant, and squaring functions.

$f(x) = ax + b$ Linear function

$f(x) = c$ Constant function

$f(x) = x^2$ Squaring function

These are examples of **polynomial functions.**

Definition of a Polynomial Function

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

is a **polynomial function of x with degree n.**

Polynomial functions are classified by degree. For example, a constant function $f(x) = c$ with $c \neq 0$ has degree 0, and a linear function $f(x) = ax + b$ with $a \neq 0$ has degree 1. In this section, you will study **quadratic functions,** which are second-degree polynomial functions.

For example, each function listed below is a quadratic function.

$f(x) = x^2 + 6x + 2$

$g(x) = 2(x + 1)^2 - 3$

$h(x) = 9 + \frac{1}{4}x^2$

$k(x) = (x - 2)(x + 1)$

Note that the squaring function is a simple quadratic function.

Definition of a Quadratic Function

Let a, b, and c be real numbers with $a \neq 0$. The function

$f(x) = ax^2 + bx + c$ Quadratic function

is a **quadratic function.**

Time, t	Height, h
0	6
4	774
8	1030
12	774
16	6

Often, quadratic functions can model real-life data. For example, the table at the left shows the heights h (in feet) of a projectile fired from an initial height of 6 feet with an initial velocity of 256 feet per second at selected values of time t (in seconds). A quadratic model for the data in the table is

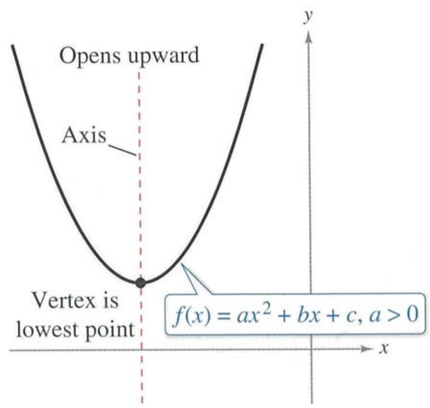
$$h(t) = -16t^2 + 256t + 6, \quad 0 \le t \le 16.$$

The graph of a quadratic function is a "U"-shaped curve called a **parabola.** Parabolas occur in many real-life applications—including those that involve reflective properties of satellite dishes and flashlight reflectors. You will study these properties in Section 10.2.

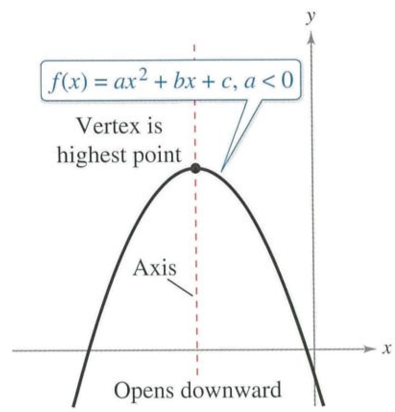
All parabolas are symmetric with respect to a line called the **axis of symmetry,** or simply the **axis** of the parabola. The point where the axis intersects the parabola is the **vertex** of the parabola. When the leading coefficient is positive, the graph of

$$f(x) = ax^2 + bx + c$$

is a parabola that opens upward. When the leading coefficient is negative, the graph is a parabola that opens downward. The next two figures show the axes and vertices of parabolas for cases where $a > 0$ and $a < 0$.

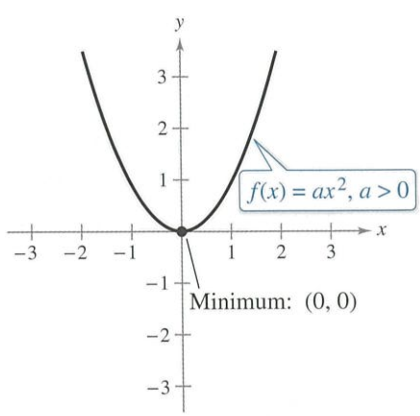


Leading coefficient is positive.

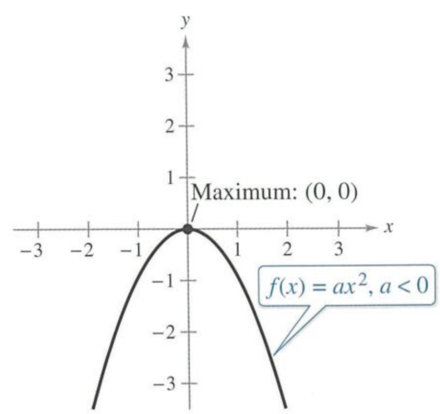


Leading coefficient is negative.

The simplest type of quadratic function is one in which $b = c = 0$. In this case, the function has the form $f(x) = ax^2$. Its graph is a parabola whose vertex is $(0, 0)$. When $a > 0$, the vertex is the point with the *minimum* y-value on the graph, and when $a < 0$, the vertex is the point with the *maximum* y-value on the graph, as shown in the figures below.



Leading coefficient is positive.



Leading coefficient is negative.

When sketching the graph of $f(x) = ax^2$, it is helpful to use the graph of $y = x^2$ as a reference, as suggested in Section 1.7. There you learned that when $a > 1$, the graph of $y = af(x)$ is a vertical stretch of the graph of $y = f(x)$. When $0 < a < 1$, the graph of $y = af(x)$ is a vertical shrink of the graph of $y = f(x)$. Example 1 demonstrates this again.

EXAMPLE 1 **Sketching Graphs of Quadratic Functions**

See LarsonPrecalculus.com for an interactive version of this type of example.

Sketch the graph of each quadratic function and compare it with the graph of $y = x^2$.

a. $f(x) = \frac{1}{3}x^2$ **b.** $g(x) = 2x^2$

> **ALGEBRA HELP** To review techniques for shifting, reflecting, stretching, and shrinking graphs, see Section 1.7.

Solution

a. Compared with $y = x^2$, each output of $f(x) = \frac{1}{3}x^2$ "shrinks" by a factor of $\frac{1}{3}$, producing the broader parabola shown in Figure 2.1.

b. Compared with $y = x^2$, each output of $g(x) = 2x^2$ "stretches" by a factor of 2, producing the narrower parabola shown in Figure 2.2.

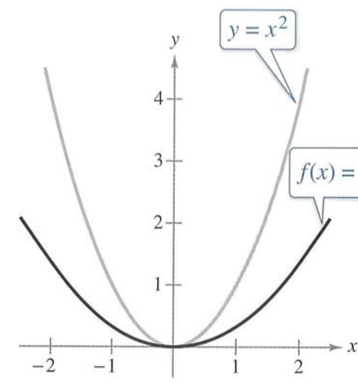


Figure 2.1

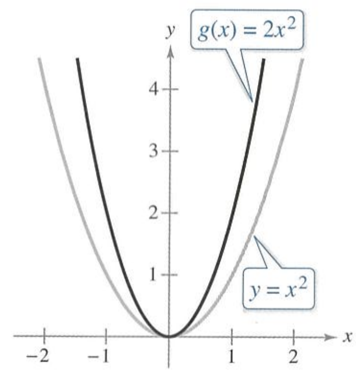


Figure 2.2

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Sketch the graph of each quadratic function and compare it with the graph of $y = x^2$.

a. $f(x) = \frac{1}{4}x^2$ **b.** $g(x) = -\frac{1}{6}x^2$ **c.** $h(x) = \frac{5}{2}x^2$ **d.** $k(x) = -4x^2$

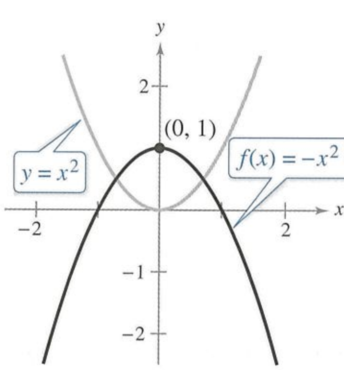
In Example 1, note that the coefficient a determines how wide the parabola $f(x) = ax^2$ opens. The smaller the value of $|a|$, the wider the parabola opens.

Recall from Section 1.7 that the graphs of

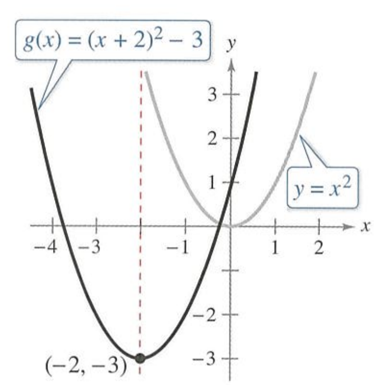
$$y = f(x \pm c), \quad y = f(x) \pm c, \quad y = f(-x), \quad \text{and} \quad y = -f(x)$$

are rigid transformations of the graph of $y = f(x)$. For example, in the figures below, notice how transformations of the graph of $y = x^2$ can produce the graphs of

$$f(x) = -x^2 + 1 \quad \text{and} \quad g(x) = (x + 2)^2 - 3.$$



Reflection in x-axis followed by an upward shift of one unit



Left shift of two units followed by a downward shift of three units

The Standard Form of a Quadratic Function

The **standard form** of a quadratic function is $f(x) = a(x - h)^2 + k$. This form is especially convenient for sketching a parabola because it identifies the vertex of the parabola as (h, k).

REMARK The standard form of a quadratic function identifies four basic transformations of the graph of $y = x^2$.

a. The factor a produces a vertical stretch or shrink.

b. When $a < 0$, the factor a also produces a reflection in the x-axis.

c. The factor $(x - h)^2$ represents a horizontal shift of h units.

d. The term k represents a vertical shift of k units.

> **Standard Form of a Quadratic Function**
>
> The quadratic function
>
> $$f(x) = a(x - h)^2 + k, \quad a \neq 0$$
>
> is in **standard form.** The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). When $a > 0$, the parabola opens upward, and when $a < 0$, the parabola opens downward.

To graph a parabola, it is helpful to begin by writing the quadratic function in standard form using the process of completing the square, as illustrated in Example 2. In this example, notice that when completing the square, you *add and subtract* the square of half the coefficient of x within the parentheses instead of adding the value to each side of the equation as is done in Appendix A.5.

EXAMPLE 2 Using Standard Form to Graph a Parabola

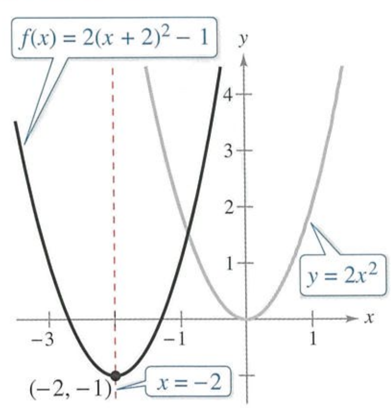
Sketch the graph of $f(x) = 2x^2 + 8x + 7$. Identify the vertex and the axis of the parabola.

ALGEBRA HELP To review techniques for completing the square, see Appendix A.5.

Solution Begin by writing the quadratic function in standard form. Notice that the first step in completing the square is to factor out any coefficient of x^2 that is not 1.

$$\begin{aligned}
f(x) &= 2x^2 + 8x + 7 && \text{Write original function.}\\
&= 2(x^2 + 4x) + 7 && \text{Factor 2 out of } x\text{-terms.}\\
&= 2(x^2 + 4x + 4 - 4) + 7 && \text{Add and subtract 4 within parentheses.}\\
&\qquad (4/2)^2 &&\\
&= 2(x^2 + 4x + 4) - 2(4) + 7 && \text{Distributive Property}\\
&= 2(x^2 + 4x + 4) - 8 + 7 && \text{Simplify.}\\
&= 2(x + 2)^2 - 1 && \text{Write in standard form.}
\end{aligned}$$

The graph of f is a parabola that opens upward and has its vertex at $(-2, -1)$. This corresponds to a left shift of two units and a downward shift of one unit relative to the graph of $y = 2x^2$, as shown in the figure. The axis of the parabola is the vertical line through the vertex, $x = -2$, also shown in the figure.



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Sketch the graph of $f(x) = 3x^2 - 6x + 4$. Identify the vertex and the axis of the parabola.

ALGEBRA HELP To review techniques for solving quadratic equations, see Appendix A.5.

To find the x-intercepts of the graph of $f(x) = ax^2 + bx + c$, you must solve the equation $ax^2 + bx + c = 0$. When $ax^2 + bx + c$ does not factor, use completing the square or the Quadratic Formula to find the x-intercepts. Remember, however, that a parabola may not have x-intercepts.

EXAMPLE 3 Finding the Vertex and x-Intercepts of a Parabola

Sketch the graph of $f(x) = -x^2 + 6x - 8$. Identify the vertex and x-intercepts.

Solution

$$f(x) = -x^2 + 6x - 8 \quad \text{Write original function.}$$

$$= -(x^2 - 6x) - 8 \quad \text{Factor } -1 \text{ out of } x\text{-terms.}$$

$$= -(x^2 - 6x + 9 - 9) - 8 \quad \text{Add and subtract 9 within parentheses.}$$

$$(-6/2)^2$$

$$= -(x^2 - 6x + 9) - (-9) - 8 \quad \text{Distributive Property}$$

$$= -(x - 3)^2 + 1 \quad \text{Write in standard form.}$$

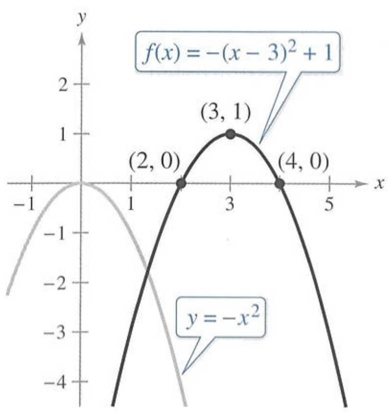

Figure 2.3

The graph of f is a parabola that opens downward with vertex $(3, 1)$. Next, find the x-intercepts of the graph.

$$-(x^2 - 6x + 8) = 0 \quad \text{Factor out } -1.$$

$$-(x - 2)(x - 4) = 0 \quad \text{Factor.}$$

$$x - 2 = 0 \Rightarrow x = 2 \quad \text{Set 1st factor equal to 0 and solve.}$$

$$x - 4 = 0 \Rightarrow x = 4 \quad \text{Set 2nd factor equal to 0 and solve.}$$

So, the x-intercepts are $(2, 0)$ and $(4, 0)$, as shown in Figure 2.3.

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Sketch the graph of $f(x) = x^2 - 4x + 3$. Identify the vertex and x-intercepts.

EXAMPLE 4 Writing a Quadratic Function

Write the standard form of the quadratic function whose graph is a parabola with vertex $(1, 2)$ and that passes through the point $(3, -6)$.

Solution The vertex is $(h, k) = (1, 2)$, so the equation has the form

$$f(x) = a(x - 1)^2 + 2. \quad \text{Substitute for } h \text{ and } k \text{ in standard form.}$$

The parabola passes through the point $(3, -6)$, so it follows that $f(3) = -6$. So,

$$f(x) = a(x - 1)^2 + 2 \quad \text{Write in standard form.}$$

$$-6 = a(3 - 1)^2 + 2 \quad \text{Substitute 3 for } x \text{ and } -6 \text{ for } f(x).$$

$$-6 = 4a + 2 \quad \text{Simplify.}$$

$$-8 = 4a \quad \text{Subtract 2 from each side.}$$

$$-2 = a. \quad \text{Divide each side by 4.}$$

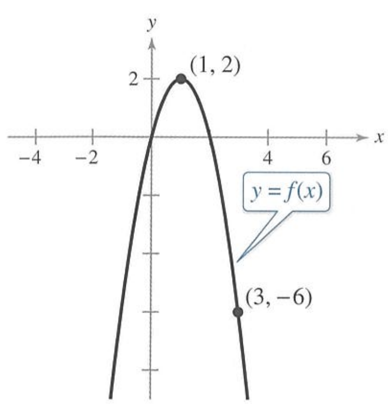

Figure 2.4

The function in standard form is $f(x) = -2(x - 2)^2 + 2$. Figure 2.4 shows the graph of f.

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Write the standard form of the quadratic function whose graph is a parabola with vertex $(-4, 11)$ and that passes through the point $(-6, 15)$.

Finding Minimum and Maximum Values

Many applications involve finding the maximum or minimum value of a quadratic function. By completing the square within the quadratic function $f(x) = ax^2 + bx + c$, you can rewrite the function in standard form (see Exercise 79).

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \qquad \text{Standard form}$$

So, the vertex of the graph of f is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Minimum and Maximum Values of Quadratic Functions

Consider the function $f(x) = ax^2 + bx + c$ with vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

1. When $a > 0$, f has a *minimum* at $x = -\frac{b}{2a}$. The minimum value is $f\left(-\frac{b}{2a}\right)$.
2. When $a < 0$, f has a *maximum* at $x = -\frac{b}{2a}$. The maximum value is $f\left(-\frac{b}{2a}\right)$.

EXAMPLE 5 Maximum Height of a Baseball

The path of a baseball after being hit is modeled by $f(x) = -0.0032x^2 + x + 3$, where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from home plate (in feet). What is the maximum height of the baseball?

Algebraic Solution

For this quadratic function, you have

$$f(x) = ax^2 + bx + c = -0.0032x^2 + x + 3$$

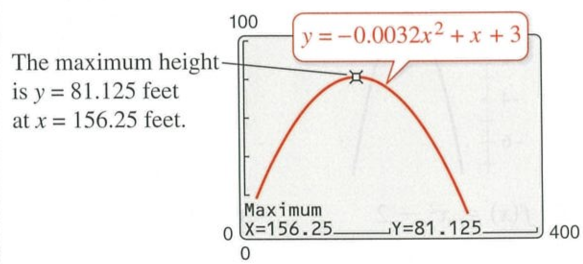
which shows that $a = -0.0032$ and $b = 1$. Because $a < 0$, the function has a maximum at $x = -b/(2a)$. So, the baseball reaches its maximum height when it is

$$x = -\frac{b}{2a} = -\frac{1}{2(-0.0032)} = 156.25 \text{ feet}$$

from home plate. At this distance, the maximum height is

$$f(156.25) = -0.0032(156.25)^2 + 156.25 + 3 = 81.125 \text{ feet.}$$

Graphical Solution



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Rework Example 5 when the path of the baseball is modeled by

$$f(x) = -0.007x^2 + x + 4.$$

Summarize (Section 2.1)

1. State the definition of a quadratic function and describe its graph *(pages 114–116)*. For an example of sketching graphs of quadratic functions, see Example 1.
2. State the standard form of a quadratic function *(page 117)*. For examples that use the standard form of a quadratic function, see Examples 2–4.
3. Explain how to find the minimum or maximum value of a quadratic function *(page 119)*. For a real-life application, see Example 5.

2.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

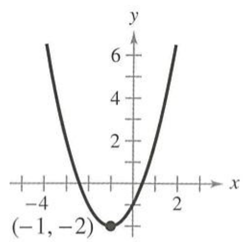
Vocabulary: Fill in the blanks.

1. Linear, constant, and squaring functions are examples of ________ functions.
2. A polynomial function of x with degree n has the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0$ $(a_n \neq 0)$, where n is a ________ ________ and $a_n, a_{n-1}, \ldots, a_1, a_0$ are ________ numbers.
3. A ________ function is a second-degree polynomial function, and its graph is called a ________.
4. When the graph of a quadratic function opens downward, its leading coefficient is ________ and the vertex of the graph is a ________.

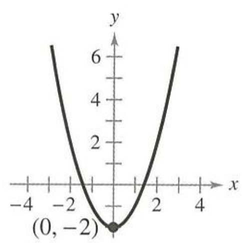
Skills and Applications

Matching **In Exercises 5–8, match the quadratic function with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

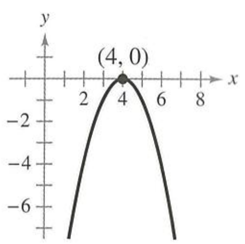
(a)



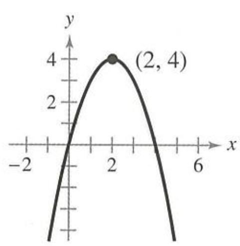
(b)



(c)



(d)



5. $f(x) = x^2 - 2$
6. $f(x) = (x + 1)^2 - 2$
7. $f(x) = -(x - 4)^2$
8. $f(x) = 4 - (x - 2)^2$

Sketching Graphs of Quadratic Functions **In Exercises 9–12, sketch the graph of each quadratic function and compare it with the graph of $y = x^2$.**

9. (a) $f(x) = \frac{1}{2}x^2$ (b) $g(x) = -\frac{1}{8}x^2$
   (c) $h(x) = \frac{3}{2}x^2$ (d) $k(x) = -3x^2$
10. (a) $f(x) = x^2 + 1$ (b) $g(x) = x^2 - 1$
    (c) $h(x) = x^2 + 3$ (d) $k(x) = x^2 - 3$
11. (a) $f(x) = (x - 1)^2$ (b) $g(x) = (3x)^2 + 1$
    (c) $h(x) = \left(\frac{1}{3}x\right)^2 - 3$ (d) $k(x) = (x + 3)^2$
12. (a) $f(x) = -\frac{1}{2}(x - 2)^2 + 1$
    (b) $g(x) = \left[\frac{1}{2}(x - 1)\right]^2 - 3$
    (c) $h(x) = -\frac{1}{2}(x + 2)^2 - 1$
    (d) $k(x) = [2(x + 1)]^2 + 4$

Using Standard Form to Graph a Parabola **In Exercises 13–26, write the quadratic function in standard form and sketch its graph. Identify the vertex, axis of symmetry, and x-intercept(s).**

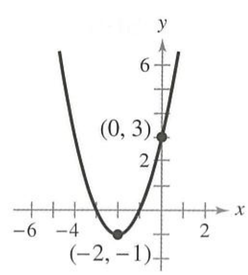
13. $f(x) = x^2 - 6x$
14. $g(x) = x^2 - 8x$
15. $h(x) = x^2 - 8x + 16$
16. $g(x) = x^2 + 2x + 1$
17. $f(x) = x^2 - 6x + 2$
18. $f(x) = x^2 + 16x + 61$
19. $f(x) = x^2 - 8x + 21$
20. $f(x) = x^2 + 12x + 40$
21. $f(x) = x^2 - x + \frac{5}{4}$
22. $f(x) = x^2 + 3x + \frac{1}{4}$
23. $f(x) = -x^2 + 2x + 5$
24. $f(x) = -x^2 - 4x + 1$
25. $h(x) = 4x^2 - 4x + 21$
26. $f(x) = 2x^2 - x + 1$

Using Technology **In Exercises 27–34, use a graphing utility to graph the quadratic function. Identify the vertex, axis of symmetry, and x-intercept(s). Then check your results algebraically by writing the quadratic function in standard form.**

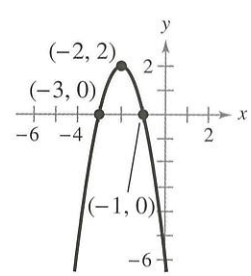
27. $f(x) = -(x^2 + 2x - 3)$
28. $f(x) = -(x^2 + x - 30)$
29. $g(x) = x^2 + 8x + 11$
30. $f(x) = x^2 + 10x + 14$
31. $f(x) = -2x^2 + 12x - 18$
32. $f(x) = -4x^2 + 24x - 41$
33. $g(x) = \frac{1}{2}(x^2 + 4x - 2)$
34. $f(x) = \frac{3}{5}(x^2 + 6x - 5)$

Writing a Quadratic Function **In Exercises 35 and 36, write the standard form of the quadratic function whose graph is the parabola shown.**

35.



36.

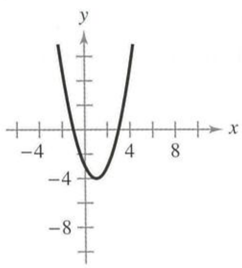


Writing a Quadratic Function In Exercises 37–46, write the standard form of the quadratic function whose graph is a parabola with the given vertex and that passes through the given point.

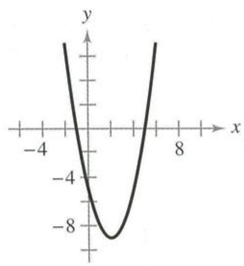
37. Vertex: $(-2, 5)$; point: $(0, 9)$
38. Vertex: $(-3, -10)$; point: $(0, 8)$
39. Vertex: $(1, -2)$; point: $(-1, 14)$
40. Vertex: $(2, 3)$; point: $(0, 2)$
41. Vertex: $(5, 12)$; point: $(7, 15)$
42. Vertex: $(-2, -2)$; point: $(-1, 0)$
43. Vertex: $\left(-\frac{1}{4}, \frac{3}{2}\right)$; point: $(-2, 0)$
44. Vertex: $\left(\frac{5}{2}, -\frac{3}{4}\right)$; point: $(-2, 4)$
45. Vertex: $\left(-\frac{5}{2}, 0\right)$; point: $\left(-\frac{7}{2}, -\frac{16}{3}\right)$
46. Vertex: $(6, 6)$; point: $\left(\frac{61}{10}, \frac{3}{2}\right)$

Graphical Reasoning In Exercises 47–50, determine the x-intercept(s) of the graph visually. Then find the x-intercept(s) algebraically to confirm your results.

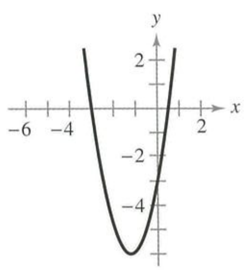
47. $y = x^2 - 2x - 3$



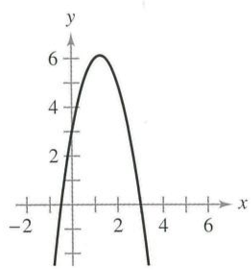
48. $y = x^2 - 4x - 5$



49. $y = 2x^2 + 5x - 3$



50. $y = -2x^2 + 5x + 3$



Using Technology In Exercises 51–56, use a graphing utility to graph the quadratic function. Find the x-intercept(s) of the graph and compare them with the solutions of the corresponding quadratic equation when $f(x) = 0$.

51. $f(x) = x^2 - 4x$
52. $f(x) = -2x^2 + 10x$
53. $f(x) = x^2 - 9x + 18$
54. $f(x) = x^2 - 8x - 20$
55. $f(x) = 2x^2 - 7x - 30$
56. $f(x) = \frac{7}{10}(x^2 + 12x - 45)$

Finding Quadratic Functions In Exercises 57–62, find two quadratic functions, one that opens upward and one that opens downward, whose graphs have the given x-intercepts. (There are many correct answers.)

57. $(-3, 0), (3, 0)$
58. $(-5, 0), (5, 0)$
59. $(-1, 0), (4, 0)$
60. $(-2, 0), (3, 0)$
61. $(-3, 0), \left(-\frac{1}{2}, 0\right)$
62. $\left(-\frac{3}{2}, 0\right), (-5, 0)$

Number Problems In Exercises 63–66, find two positive real numbers whose product is a maximum.

63. The sum is 110.
64. The sum is S.
65. The sum of the first and twice the second is 24.
66. The sum of the first and three times the second is 42.

67. **Path of a Diver**

The path of a diver is modeled by

$$f(x) = -\frac{4}{9}x^2 + \frac{24}{9}x + 12$$

where $f(x)$ is the height (in feet) and x is the horizontal distance (in feet) from the end of the diving board. What is the maximum height of the diver?

68. **Height of a Ball** The path of a punted football is modeled by

$$f(x) = -\frac{16}{2025}x^2 + \frac{9}{5}x + 1.5$$

where $f(x)$ is the height (in feet) and x is the horizontal distance (in feet) from the point at which the ball is punted.

(a) How high is the ball when it is punted?

(b) What is the maximum height of the punt?

(c) How long is the punt?

69. **Minimum Cost** A manufacturer of lighting fixtures has daily production costs of $C = 800 - 10x + 0.25x^2$, where C is the total cost (in dollars) and x is the number of units produced. What daily production number yields a minimum cost?

70. **Maximum Profit** The profit P (in hundreds of dollars) that a company makes depends on the amount x (in hundreds of dollars) the company spends on advertising according to the model $P = 230 + 20x - 0.5x^2$. What expenditure for advertising yields a maximum profit?

71. Maximum Revenue The total revenue R earned (in thousands of dollars) from manufacturing handheld video games is given by $R(p) = -25p^2 + 1200p$, where p is the price per unit (in dollars).

(a) Find the revenues when the prices per unit are \$20, \$25, and \$30.

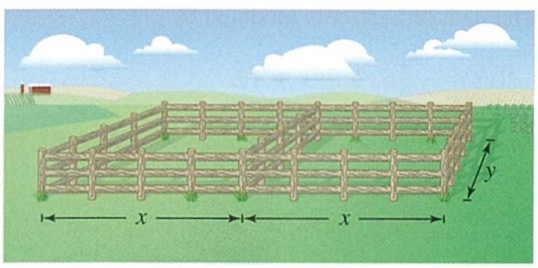
(b) Find the unit price that yields a maximum revenue. What is the maximum revenue? Explain.

72. Maximum Revenue The total revenue R earned per day (in dollars) from a pet-sitting service is given by $R(p) = -12p^2 + 150p$, where p is the price charged per pet (in dollars).

(a) Find the revenues when the prices per pet are \$4, \$6, and \$8.

(b) Find the unit price that yields a maximum revenue. What is the maximum revenue? Explain.

73. Maximum Area A rancher has 200 feet of fencing to enclose two adjacent rectangular corrals (see figure).



(a) Write the area A of the corrals as a function of x.

(b) What dimensions produce a maximum enclosed area?

74. Maximum Area A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window (see figure). The perimeter of the window is 16 feet.

(a) Write the area A of the window as a function of x.

(b) What dimensions produce a window of maximum area?

Exploration

True or False? **In Exercises 75 and 76, determine whether the statement is true or false. Justify your answer.**

75. The graph of $f(x) = -12x^2 - 1$ has no x-intercepts.

76. The graphs of $f(x) = -4x^2 - 10x + 7$ and $g(x) = 12x^2 + 30x + 1$ have the same axis of symmetry.

Think About It **In Exercises 77 and 78, find the values of b such that the function has the given maximum or minimum value.**

77. $f(x) = -x^2 + bx - 75$; Maximum value: 25

78. $f(x) = x^2 + bx - 25$; Minimum value: -50

79. Verifying the Vertex Write the quadratic function

$$f(x) = ax^2 + bx + c$$

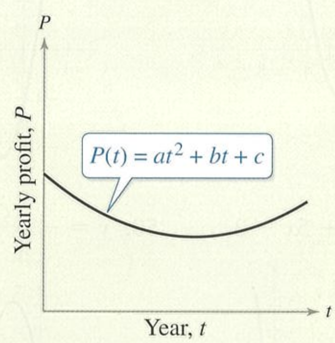
in standard form to verify that the vertex occurs at

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).$$

80. HOW DO YOU SEE IT? The graph shows a quadratic function of the form

$$P(t) = at^2 + bt + c$$

which represents the yearly profit for a company, where $P(t)$ is the profit in year t.



(a) Is the value of a positive, negative, or zero? Explain.

(b) Write an expression in terms of a and b that represents the year t when the company made the least profit.

(c) The company made the same yearly profits in 2008 and 2016. Estimate the year in which the company made the least profit.

81. Proof Assume that the function

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

has two real zeros. Prove that the x-coordinate of the vertex of the graph is the average of the zeros of f. (*Hint:* Use the Quadratic Formula.)

Project: Height of a Basketball To work an extended application analyzing the height of a dropped basketball, visit this text's website at *LarsonPrecalculus.com.*

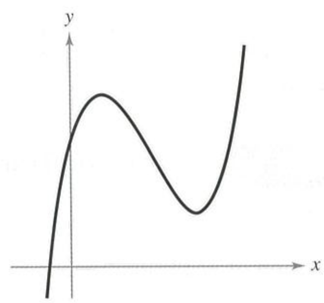
2.2 Polynomial Functions of Higher Degree

Polynomial functions have many real-life applications. For example, in Exercise 98 on page 135, you will use a polynomial function to analyze the growth of a red oak tree.

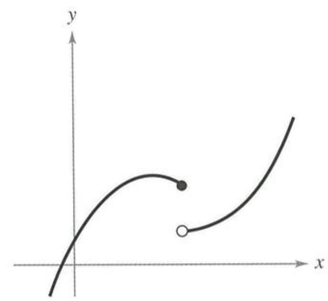
- Use transformations to sketch graphs of polynomial functions.
- Use the Leading Coefficient Test to determine the end behaviors of graphs of polynomial functions.
- Find real zeros of polynomial functions and use them as sketching aids.
- Use the Intermediate Value Theorem to help locate real zeros of polynomial functions.

Graphs of Polynomial Functions

In this section, you will study basic features of the graphs of polynomial functions. One feature is that the graph of a polynomial function is **continuous.** Essentially, this means that the graph of a polynomial function has no breaks, holes, or gaps, as shown in Figure 2.5(a). The graph shown in Figure 2.5(b) is an example of a piecewise-defined function that is not continuous.



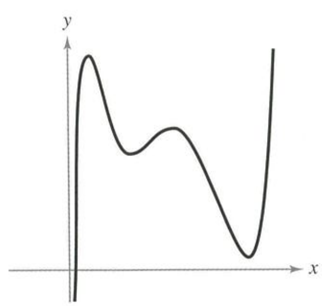
(a) Polynomial functions have continuous graphs.



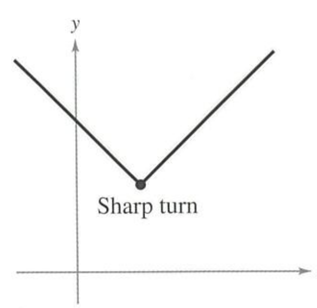
(b) Functions with graphs that are not continuous are not polynomial functions.

Figure 2.5

Another feature of the graph of a polynomial function is that it has only smooth, rounded turns, as shown in Figure 2.6(a). The graph of a polynomial function cannot have a sharp turn, such as the one shown in Figure 2.6(b).



(a) Polynomial functions have graphs with smooth, rounded turns.



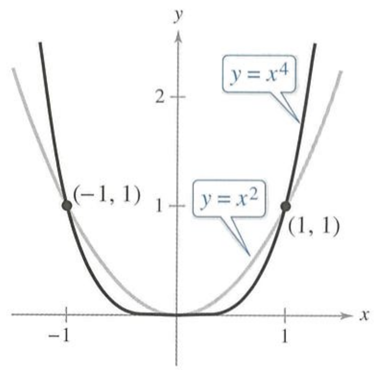
(b) Functions with graphs that have sharp turns are not polynomial functions.

Figure 2.6

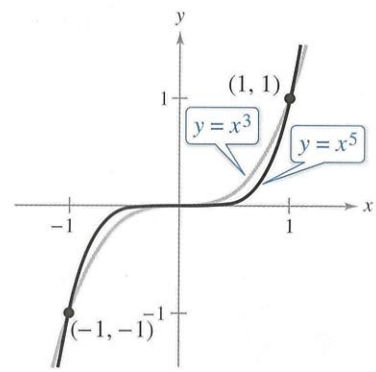
Sketching graphs of polynomial functions of degree greater than 2 is often more involved than sketching graphs of polynomial functions of degree 0, 1, or 2. However, using the features presented in this section, along with your knowledge of point plotting, intercepts, and symmetry, you should be able to make reasonably accurate sketches by hand.

REMARK For functions of the form $f(x) = x^n$, if n is even, then the graph of the function is symmetric with respect to the y-axis, and if n is odd, then the graph of the function is symmetric with respect to the origin.

The polynomial functions that have the simplest graphs are monomial functions of the form $f(x) = x^n$, where n is an integer greater than zero. When n is *even*, the graph is similar to the graph of $f(x) = x^2$, and when n is *odd*, the graph is similar to the graph of $f(x) = x^3$, as shown in Figure 2.7. Moreover, the greater the value of n, the flatter the graph near the origin. Polynomial functions of the form $f(x) = x^n$ are often referred to as **power functions.**



(a) When n is even, the graph of $y = x^n$ touches the x-axis at the x-intercept.



(b) When n is odd, the graph of $y = x^n$ crosses the x-axis at the x-intercept.

Figure 2.7

EXAMPLE 1 Sketching Transformations of Monomial Functions

See LarsonPrecalculus.com for an interactive version of this type of example.

Sketch the graph of each function.

a. $f(x) = -x^5$ **b.** $h(x) = (x + 1)^4$

Solution

a. The degree of $f(x) = -x^5$ is odd, so its graph is similar to the graph of $y = x^3$. In Figure 2.8, note that the negative coefficient has the effect of reflecting the graph in the x-axis.

b. The degree of $h(x) = (x + 1)^4$ is even, so its graph is similar to the graph of $y = x^2$. In Figure 2.9, note that the graph of h is a left shift by one unit of the graph of $y = x^4$.

ALGEBRA HELP To review techniques for shifting, reflecting, stretching, and shrinking graphs, see Section 1.7.

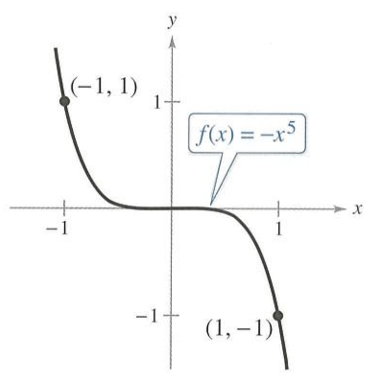


Figure 2.8

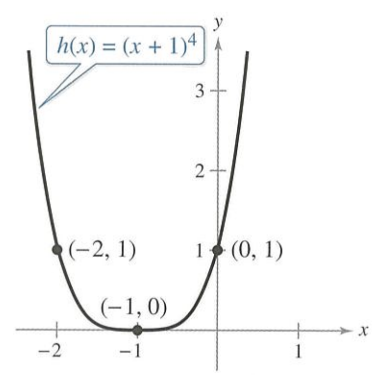


Figure 2.9

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Sketch the graph of each function.

a. $f(x) = (x + 5)^4$ **b.** $g(x) = x^4 - 7$

c. $h(x) = 7 - x^4$ **d.** $k(x) = \frac{1}{4}(x - 3)^4$

The Leading Coefficient Test

In Example 1, note that both graphs eventually rise or fall without bound as x moves to the left or to the right. A polynomial function's degree (even or odd) and its leading coefficient (positive or negative) determine whether the graph of the function eventually rises or falls, as described in the **Leading Coefficient Test.**

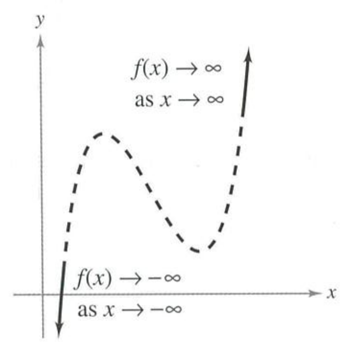
Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function

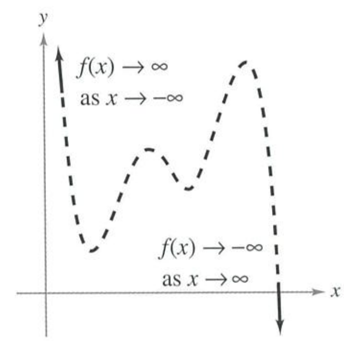
$$f(x) = a_n x^n + \cdot \cdot \cdot + a_1 x + a_0, \quad a_n \neq 0$$

eventually rises or falls in the manner described below.

1. When n is *odd:*

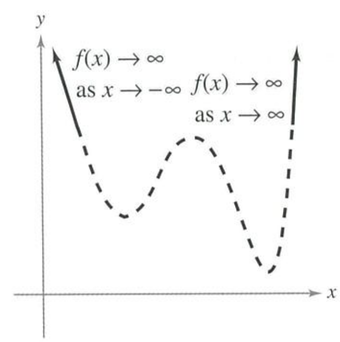


If the leading coefficient is positive ($a_n > 0$), then the graph falls to the left and rises to the right.

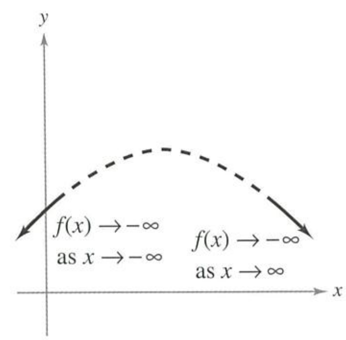


If the leading coefficient is negative ($a_n < 0$), then the graph rises to the left and falls to the right.

2. When n is *even:*



If the leading coefficient is positive ($a_n > 0$), then the graph rises to the left and to the right.



If the leading coefficient is negative ($a_n < 0$), then the graph falls to the left and to the right.

The dashed portions of the graphs indicate that the test determines *only* the right-hand and left-hand behavior of the graph.

REMARK The notation "$f(x) \rightarrow -\infty$ as $x \rightarrow -\infty$" means that the graph falls to the left. The notation "$f(x) \rightarrow \infty$ as $x \rightarrow \infty$" means that the graph rises to the right. Identify and interpret similar notation for the other two possible types of end behavior given in the Leading Coefficient Test.

As you continue to study polynomial functions and their graphs, you will notice that the degree of a polynomial plays an important role in determining other characteristics of the polynomial function and its graph.

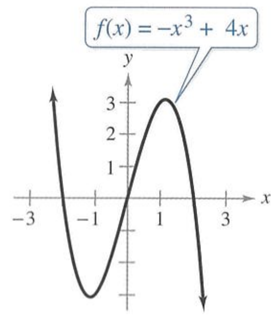
EXAMPLE 2 Applying the Leading Coefficient Test

Describe the left-hand and right-hand behavior of the graph of each function.

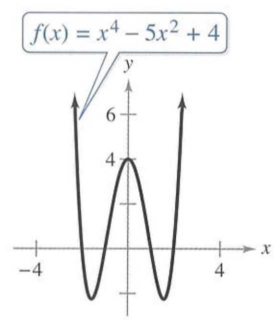
a. $f(x) = -x^3 + 4x$ **b.** $f(x) = x^4 - 5x^2 + 4$ **c.** $f(x) = x^5 - x$

Solution

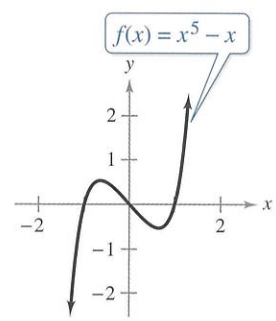
a. The degree is odd and the leading coefficient is negative, so the graph rises to the left and falls to the right, as shown in the figure below.



b. The degree is even and the leading coefficient is positive, so the graph rises to the left and to the right, as shown in the figure below.



c. The degree is odd and the leading coefficient is positive, so the graph falls to the left and rises to the right, as shown in the figure below.



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Describe the left-hand and right-hand behavior of the graph of each function.

a. $f(x) = \frac{1}{4}x^3 - 2x$ **b.** $f(x) = -3.6x^5 + 5x^3 - 1$

In Example 2, note that the Leading Coefficient Test tells you only whether the graph *eventually* rises or falls to the left or to the right. You must use other tests to determine other characteristics of the graph, such as intercepts and minimum and maximum points.

Real Zeros of Polynomial Functions

It is possible to show that for a polynomial function f of degree n, the two statements below are true.

1. The function f has, at most, n real zeros. (You will study this result in detail in the discussion of the Fundamental Theorem of Algebra in Section 2.5.)
2. The graph of f has, at most, $n - 1$ turning points. (Turning points, also called relative minima or relative maxima, are points at which the graph changes from increasing to decreasing or vice versa.)

REMARK Remember that the *zeros* of a function of x are the x-values for which the function is zero.

Finding the zeros of a polynomial function is an important problem in algebra. There is a strong interplay between graphical and algebraic approaches to this problem.

> **Real Zeros of Polynomial Functions**
>
> When f is a polynomial function and a is a real number, the statements listed below are equivalent.
>
> 1. $x = a$ is a *zero* of the function f.
> 2. $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.
> 3. $(x - a)$ is a *factor* of the polynomial $f(x)$.
> 4. $(a, 0)$ is an *x-intercept* of the graph of f.

EXAMPLE 3 Finding Real Zeros of a Polynomial Function

Find all real zeros of $f(x) = -2x^4 + 2x^2$. Then determine the maximum possible number of turning points of the graph of the function.

Solution To find the real zeros of the function, set $f(x)$ equal to zero and then solve for x.

$$-2x^4 + 2x^2 = 0 \quad \text{Set } f(x) \text{ equal to 0.}$$
$$-2x^2(x^2 - 1) = 0 \quad \text{Remove common monomial factor.}$$
$$-2x^2(x - 1)(x + 1) = 0 \quad \text{Factor completely.}$$

So, the real zeros are $x = 0$, $x = 1$, and $x = -1$, and the corresponding x-intercepts occur at $(0, 0)$, $(1, 0)$, and $(-1, 0)$. The function is a fourth-degree polynomial, so the graph of f can have at most $4 - 1 = 3$ turning points. In this case, the graph of f has three turning points. Figure 2.10 shows the graph of f.

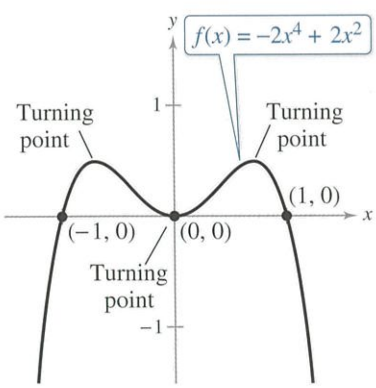

Figure 2.10

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Find all real zeros of $f(x) = x^3 - 12x^2 + 36x$. Then determine the maximum possible number of turning points of the graph of the function.

ALGEBRA HELP The solution to Example 3 uses polynomial factoring. To review the techniques for factoring polynomials, see Appendix A.3.

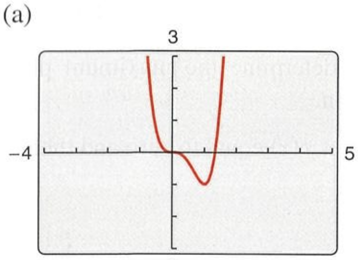
In Example 3, note that the factor $-2x^2$ yields the *repeated* zero $x = 0$. The exponent is even, so the graph touches the x-axis at $x = 0$.

> **Repeated Zeros**
>
> A factor $(x - a)^k$, $k > 1$, yields a **repeated zero** $x = a$ of **multiplicity** k.
>
> 1. When k is odd, the graph *crosses* the x-axis at $x = a$.
> 2. When k is even, the graph *touches* the x-axis (but does not cross the x-axis) at $x = a$.

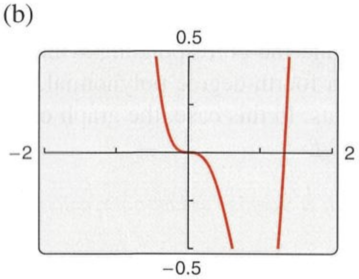
To graph polynomial functions, use the fact that a polynomial function can change signs only at its zeros. Between two consecutive zeros, a polynomial must be entirely positive or entirely negative. (This follows from the Intermediate Value Theorem, which you will study later in this section.) This means that when you put the real zeros of a polynomial function in order, they divide the real number line into intervals in which the function has no sign changes. These resulting intervals are **test intervals** in which you choose a representative x-value to determine whether the value of the polynomial function is positive (the graph lies above the x-axis) or negative (the graph lies below the x-axis).

TECHNOLOGY Example 4 uses an *algebraic approach* to describe the graph of the function. A graphing utility can complement this approach. Remember to find a viewing window that shows all significant features of the graph. For instance, viewing window (a) illustrates all of the significant features of the function in Example 4, but viewing window (b) does not.

(a)



(b)



REMARK If you are unsure of the shape of a portion of the graph of a polynomial function, then plot some additional points. For instance, in Example 4, it is helpful to plot the additional point $\left(\frac{1}{2}, -\frac{5}{16}\right)$, as shown in Figure 2.12.

EXAMPLE 4 Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = 3x^4 - 4x^3$.

Solution

1. *Apply the Leading Coefficient Test.* The leading coefficient is positive and the degree is even, so you know that the graph eventually rises to the left and to the right (see Figure 2.11).
2. *Find the Real Zeros of the Function.* Factoring $f(x) = 3x^4 - 4x^3$ as $f(x) = x^3(3x - 4)$ shows that the real zeros of f are $x = 0$ and $x = \frac{4}{3}$ (both of odd multiplicity). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{4}{3}, 0\right)$. Add these points to your graph, as shown in Figure 2.11.
3. *Plot a Few Additional Points.* Use the zeros of the polynomial to find the test intervals. In each test interval, choose a representative x-value and evaluate the polynomial function, as shown in the table

Test Interval	Representative x-Value	Value of f	Sign	Point on Graph
$(-\infty, 0)$	-1	$f(-1) = 7$	Positive	$(-1, 7)$
$\left(0, \frac{4}{3}\right)$	1	$f(1) = -1$	Negative	$(1, -1)$
$\left(\frac{4}{3}, \infty\right)$	$\frac{3}{2}$	$f\left(\frac{3}{2}\right) = \frac{27}{16}$	Positive	$\left(\frac{3}{2}, \frac{27}{16}\right)$

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 2.12. Both zeros are of odd multiplicity, so you know that the graph should cross the x-axis at $x = 0$ and $x = \frac{4}{3}$.

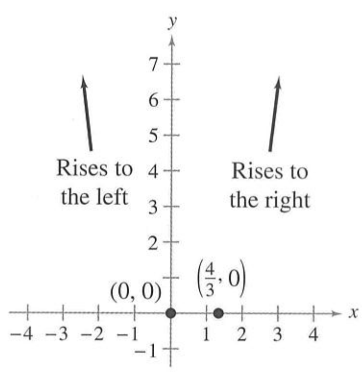


Figure 2.11

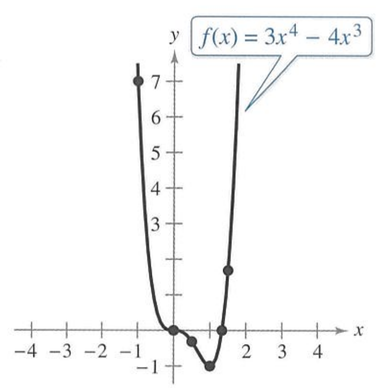


Figure 2.12

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Sketch the graph of $f(x) = 2x^3 - 6x^2$.

A polynomial function is in **standard form** when its terms are in descending order of exponents from left to right. To avoid making a mistake when applying the Leading Coefficient Test, write the polynomial function in standard form first, if necessary.

EXAMPLE 5 Sketching the Graph of a Polynomial Function

Sketch the graph of $f(x) = -\frac{9}{2}x + 6x^2 - 2x^3$.

Solution

1. *Write in Standard Form and Apply the Leading Coefficient Test.* In standard form, the polynomial function is $f(x) = -2x^3 + 6x^2 - \frac{9}{2}x$. The leading coefficient is negative and the degree is odd, so you know that the graph eventually rises to the left and falls to the right (see Figure 2.13).

2. *Find the Real Zeros of the Function.* Factoring

$$\begin{aligned} f(x) &= -2x^3 + 6x^2 - \tfrac{9}{2}x \\ &= -\tfrac{1}{2}x(4x^2 - 12x + 9) \\ &= -\tfrac{1}{2}x(2x - 3)^2 \end{aligned}$$

shows that the real zeros of f are $x = 0$ (odd multiplicity) and $x = \frac{3}{2}$ (even multiplicity). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{3}{2}, 0\right)$. Add these points to your graph, as shown in Figure 2.13.

3. *Plot a Few Additional Points.* Use the zeros of the polynomial to find the test intervals. In each test interval, choose a representative x-value and evaluate the polynomial function, as shown in the table.

Test Interval	Representative x-Value	Value of f	Sign	Point on Graph
$(-\infty, 0)$	$-\frac{1}{2}$	$f\left(-\frac{1}{2}\right) = 4$	Positive	$\left(-\frac{1}{2}, 4\right)$
$\left(0, \frac{3}{2}\right)$	$\frac{1}{2}$	$f\left(\frac{1}{2}\right) = -1$	Negative	$\left(\frac{1}{2}, -1\right)$
$\left(\frac{3}{2}, \infty\right)$	2	$f(2) = -1$	Negative	$(2, -1)$

REMARK Observe in Example 5 that the sign of $f(x)$ is positive to the left of and negative to the right of the zero $x = 0$. Similarly, the sign of $f(x)$ is negative to the left and to the right of the zero $x = \frac{3}{2}$. This illustrates that (1) if the zero of a polynomial function is of *odd* multiplicity, then the graph crosses the x-axis at that zero, and (2) if the zero is of *even* multiplicity, then the graph touches the x-axis at that zero.

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 2.14. From the multiplicities of the zeros, you know that the graph crosses the x-axis at $(0, 0)$ but does not cross the x-axis at $\left(\frac{3}{2}, 0\right)$.

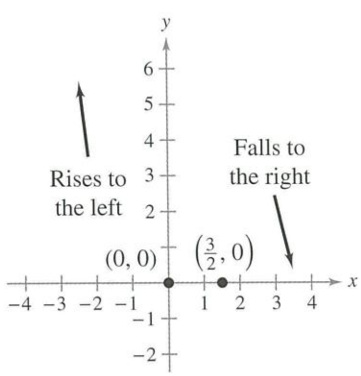


Figure 2.13

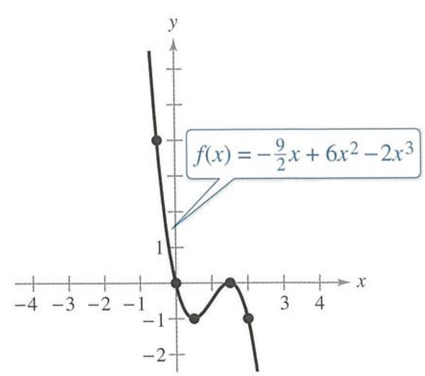


Figure 2.14

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

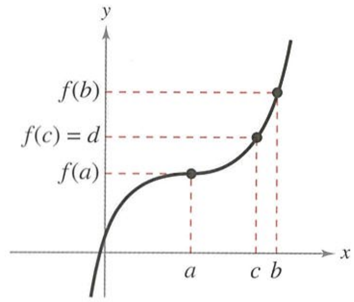
Sketch the graph of $f(x) = -\frac{1}{4}x^4 + \frac{3}{2}x^3 - \frac{9}{4}x^2$.

The Intermediate Value Theorem

The **Intermediate Value Theorem** implies that if

$$(a, f(a)) \quad \text{and} \quad (b, f(b))$$

are two points on the graph of a polynomial function such that $f(a) \neq f(b)$, then for any number d between $f(a)$ and $f(b)$ there must be a number c between a and b such that $f(c) = d$. (See figure below.)



Intermediate Value Theorem

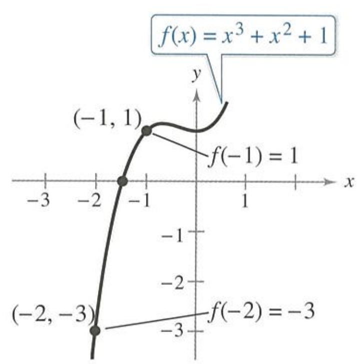
Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then, in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

One application of the Intermediate Value Theorem is in helping you locate real zeros of a polynomial function. If there exists a value $x = a$ at which a polynomial function is negative, and another value $x = b$ at which it is positive (or if it is positive when $x = a$ and negative when $x = b$), then the function has at least one real zero between these two values. For example, the function

$$f(x) = x^3 + x^2 + 1$$

is negative when $x = -2$ and positive when $x = -1$. So, it follows from the Intermediate Value Theorem that f must have a real zero somewhere between -2 and -1, as shown in the figure below.

REMARK Note that $f(a)$ and $f(b)$ must be of opposite signs in order to guarantee that a zero exists between them. If $f(a)$ and $f(b)$ are of the same sign, then it is inconclusive whether a zero exists between them.



The function f must have a real zero somewhere between -2 and -1.

By continuing this line of reasoning, it is possible to approximate real zeros of a polynomial function to any desired accuracy. Example 6 further demonstrates this concept.

▷ **TECHNOLOGY** Using the *table* feature of a graphing utility can help you approximate real zeros of polynomial functions. For instance, in Example 6, construct a table that shows function values for integer values of x. Scrolling through the table, notice that $f(-1)$ and $f(0)$ differ in sign.

X	Y1	
-2	-11	
-1	-1	
0	1	
1	1	
2	5	
3	19	
4	49	

X=0

So, by the Intermediate Value Theorem, the function has a real zero between -1 and 0. Adjust your table to show function values for $-1 \le x \le 0$ using increments of 0.1. Scrolling through this table, notice that $f(-0.8)$ and $f(-0.7)$ differ in sign.

X	Y1	
-1	-1	
-.9	-.539	
-.8	-.152	
-.7	.167	
-.6	.424	
-.5	.625	
-.4	.776	

X=-.7

So, the function has a real zero between -0.8 and -0.7. Repeating this process with smaller increments, you should obtain $x \approx -0.755$ as the real zero of the function to three decimal places, as stated in Example 6. Use the *zero* or *root* feature of the graphing utility to confirm this result.

EXAMPLE 6 Using the Intermediate Value Theorem

Use the Intermediate Value Theorem to approximate the real zero of

$$f(x) = x^3 - x^2 + 1.$$

Solution Begin by computing a few function values.

x	-2	-1	0	1
$f(x)$	-11	-1	1	1

The value $f(-1)$ is negative and $f(0)$ is positive, so by the Intermediate Value Theorem, the function has a real zero between -1 and 0. To pinpoint this zero more closely, divide the interval $[-1, 0]$ into tenths and evaluate the function at each point. When you do this, you will find that

$$f(-0.8) = -0.152$$

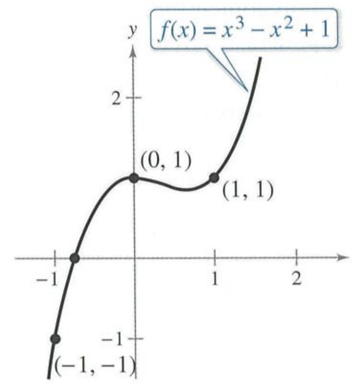
and

$$f(-0.7) = 0.167.$$

So, f must have a real zero between -0.8 and -0.7, as shown in the figure. For a more accurate approximation, compute function values between $f(-0.8)$ and $f(-0.7)$ and apply the Intermediate Value Theorem again. Continue this process to verify that

$$x \approx -0.755$$

is an approximation (to the nearest thousandth) of the real zero of f.



The function f has a real zero between -0.8 and -0.7.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Use the Intermediate Value Theorem to approximate the real zero of

$$f(x) = x^3 - 3x^2 - 2.$$

Summarize (Section 2.2)

1. Explain how to use transformations to sketch graphs of polynomial functions *(page 124)*. For an example of sketching transformations of monomial functions, see Example 1.
2. Explain how to apply the Leading Coefficient Test *(page 125)*. For an example of applying the Leading Coefficient Test, see Example 2.
3. Explain how to find real zeros of polynomial functions and use them as sketching aids *(page 127)*. For examples involving finding real zeros of polynomial functions, see Examples 3–5.
4. Explain how to use the Intermediate Value Theorem to help locate real zeros of polynomial functions *(page 130)*. For an example of using the Intermediate Value Theorem, see Example 6.

2.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

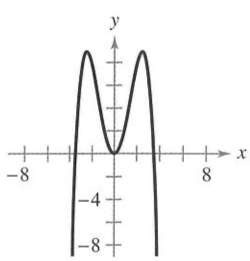
Vocabulary: Fill in the blanks.

1. The graph of a polynomial function is ________, which means that the graph has no breaks, holes, or gaps.
2. The ________ ________ ________ is used to determine the left-hand and right-hand behavior of the graph of a polynomial function.
3. A polynomial function of degree n has at most ________ real zeros and at most ________ turning points.
4. When $x = a$ is a zero of a polynomial function f, the three statements below are true.
   (a) $x = a$ is a ________ of the polynomial equation $f(x) = 0$.
   (b) ________ is a factor of the polynomial $f(x)$.
   (c) $(a, 0)$ is an ________ of the graph of f.
5. When a real zero $x = a$ of a polynomial function f is of even multiplicity, the graph of f ________ the x-axis at $x = a$, and when it is of odd multiplicity, the graph of f ________ the x-axis at $x = a$.
6. A factor $(x - a)^k$, $k > 1$, yields a ________ ________ $x = a$ of ________ k.
7. A polynomial function is written in ________ form when its terms are written in descending order of exponents from left to right.
8. The ________ ________ Theorem states that if f is a polynomial function such that $f(a) \neq f(b)$, then, in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

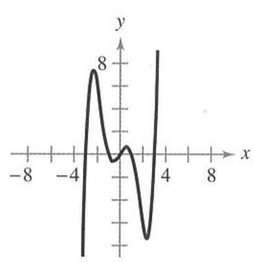
Skills and Applications

Matching **In Exercises 9–14, match the polynomial function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

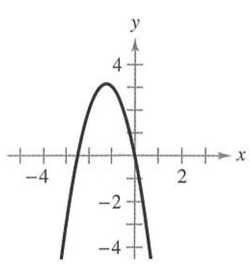
(a)



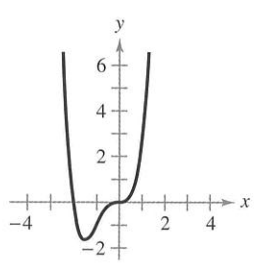
(b)



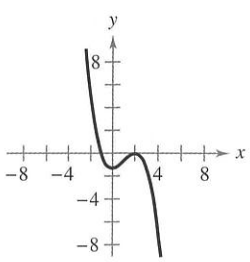
(c)



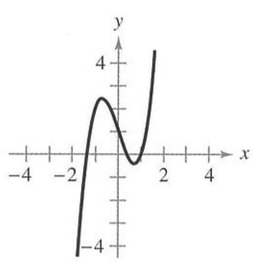
(d)



(e)



(f)



9. $f(x) = -2x^2 - 5x$
10. $f(x) = 2x^3 - 3x + 1$
11. $f(x) = -\frac{1}{4}x^4 + 3x^2$
12. $f(x) = -\frac{1}{3}x^3 + x^2 - \frac{4}{3}$
13. $f(x) = x^4 + 2x^3$
14. $f(x) = \frac{1}{5}x^5 - 2x^3 + \frac{9}{5}x$

Sketching Transformations of Monomial Functions **In Exercises 15–18, sketch the graph of $y = x^n$ and each transformation.**

15. $y = x^3$
    (a) $f(x) = (x - 4)^3$ (b) $f(x) = x^3 - 4$
    (c) $f(x) = -\frac{1}{4}x^3$ (d) $f(x) = (x - 4)^3 - 4$
16. $y = x^5$
    (a) $f(x) = (x + 1)^5$ (b) $f(x) = x^5 + 1$
    (c) $f(x) = 1 - \frac{1}{2}x^5$ (d) $f(x) = -\frac{1}{2}(x + 1)^5$
17. $y = x^4$
    (a) $f(x) = (x + 3)^4$ (b) $f(x) = x^4 - 3$
    (c) $f(x) = 4 - x^4$ (d) $f(x) = \frac{1}{2}(x - 1)^4$
    (e) $f(x) = (2x)^4 + 1$ (f) $f(x) = \left(\frac{1}{2}x\right)^4 - 2$
18. $y = x^6$
    (a) $f(x) = (x - 5)^6$ (b) $f(x) = \frac{1}{8}x^6$
    (c) $f(x) = (x + 3)^6 - 4$ (d) $f(x) = -\frac{1}{4}x^6 + 1$
    (e) $f(x) = \left(\frac{1}{4}x\right)^6 - 2$ (f) $f(x) = (2x)^6 - 1$

Applying the Leading Coefficient Test In Exercises 19–28, describe the left-hand and right-hand behavior of the graph of the polynomial function.

19. $f(x) = 12x^3 + 4x$
20. $f(x) = 2x^2 - 3x + 1$
21. $g(x) = 5 - \frac{7}{2}x - 3x^2$
22. $h(x) = 1 - x^6$
23. $h(x) = 6x - 9x^3 + x^2$
24. $g(x) = 8 + \frac{1}{4}x^5 - x^4$
25. $f(x) = 9.8x^6 - 1.2x^3$
26. $h(x) = 1 - 0.5x^5 - 2.7x^3$
27. $f(s) = -\frac{7}{8}(s^3 + 5s^2 - 7s + 1)$
28. $h(t) = -\frac{4}{3}(t - 6t^3 + 2t^4 + 9)$

Using Technology In Exercises 29–32, use a graphing utility to graph the functions f and g in the same viewing window. Zoom out sufficiently far to show that the left-hand and right-hand behaviors of f and g appear identical.

29. $f(x) = 3x^3 - 9x + 1, \quad g(x) = 3x^3$
30. $f(x) = -\frac{1}{3}(x^3 - 3x + 2), \quad g(x) = -\frac{1}{3}x^3$
31. $f(x) = -(x^4 - 4x^3 + 16x), \quad g(x) = -x^4$
32. $f(x) = 3x^4 - 6x^2, \quad g(x) = 3x^4$

Finding Real Zeros of a Polynomial Function In Exercises 33–48, (a) find all real zeros of the polynomial function, (b) determine whether the multiplicity of each zero is even or odd, (c) determine the maximum possible number of turning points of the graph of the function, and (d) use a graphing utility to graph the function and verify your answers.

33. $f(x) = x^2 - 36$
34. $f(x) = 81 - x^2$
35. $h(t) = t^2 - 6t + 9$
36. $f(x) = x^2 + 10x + 25$
37. $f(x) = \frac{1}{3}x^2 + \frac{1}{3}x - \frac{2}{3}$
38. $f(x) = \frac{1}{2}x^2 + \frac{5}{2}x - \frac{3}{2}$
39. $g(x) = 5x(x^2 - 2x - 1)$
40. $f(t) = t^2(3t^2 - 10t + 7)$
41. $f(x) = 3x^3 - 12x^2 + 3x$
42. $f(x) = x^4 - x^3 - 30x^2$
43. $g(t) = t^5 - 6t^3 + 9t$
44. $f(x) = x^5 + x^3 - 6x$
45. $f(x) = 3x^4 + 9x^2 + 6$
46. $f(t) = 2t^4 - 2t^2 - 40$
47. $g(x) = x^3 + 3x^2 - 4x - 12$
48. $f(x) = x^3 - 4x^2 - 25x + 100$

Using Technology In Exercises 49–52, (a) use a graphing utility to graph the function, (b) use the graph to approximate any x-intercepts of the graph, (c) find any real zeros of the function algebraically, and (d) compare the results of part (c) with those of part (b).

49. $y = 4x^3 - 20x^2 + 25x$
50. $y = 4x^3 + 4x^2 - 8x - 8$
51. $y = x^5 - 5x^3 + 4x$
52. $y = \frac{1}{5}x^5 - \frac{9}{5}x^3$

Finding a Polynomial Function In Exercises 53–62, find a polynomial function that has the given zeros. (There are many correct answers.)

53. $0, 7$
54. $-2, 5$
55. $0, -2, -4$
56. $0, 1, 6$
57. $4, -3, 3, 0$
58. $-2, -1, 0, 1, 2$
59. $1 + \sqrt{2}, 1 - \sqrt{2}$
60. $4 + \sqrt{3}, 4 - \sqrt{3}$
61. $2, 2 + \sqrt{5}, 2 - \sqrt{5}$
62. $3, 2 + \sqrt{7}, 2 - \sqrt{7}$

Finding a Polynomial Function In Exercises 63–70, find a polynomial of degree n that has the given zero(s). (There are many correct answers.)

	Zero(s)	Degree
63.	$x = -3$	$n = 2$
64.	$x = -\sqrt{2}, \sqrt{2}$	$n = 2$
65.	$x = -5, 0, 1$	$n = 3$
66.	$x = -2, 6$	$n = 3$
67.	$x = -5, 1, 2$	$n = 4$
68.	$x = -4, -1$	$n = 4$
69.	$x = 0, -\sqrt{3}, \sqrt{3}$	$n = 5$
70.	$x = -1, 4, 7, 8$	$n = 5$

Sketching the Graph of a Polynomial Function In Exercises 71–84, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the real zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.

71. $f(t) = \frac{1}{4}(t^2 - 2t + 15)$
72. $g(x) = -x^2 + 10x - 16$
73. $f(x) = x^3 - 25x$
74. $g(x) = -9x^2 + x^4$
75. $f(x) = -8 + \frac{1}{2}x^4$
76. $f(x) = 8 - x^3$
77. $f(x) = 3x^3 - 15x^2 + 18x$
78. $f(x) = -4x^3 + 4x^2 + 15x$
79. $f(x) = -5x^2 - x^3$
80. $f(x) = -48x^2 + 3x^4$
81. $f(x) = 9x^2(x + 2)^3$
82. $h(x) = \frac{1}{3}x^3(x - 4)^2$
83. $g(t) = -\frac{1}{4}(t - 2)^2(t + 2)^2$
84. $g(x) = \frac{1}{10}(x + 1)^2(x - 3)^3$

Using Technology In Exercises 85–88, use a graphing utility to graph the function. Use the *zero* or *root* feature to approximate the real zeros of the function. Then determine whether the multiplicity of each zero is even or odd.

85. $f(x) = x^3 - 16x$
86. $f(x) = \frac{1}{4}x^4 - 2x^2$
87. $g(x) = \frac{1}{5}(x + 1)^2(x - 3)(2x - 9)$
88. $h(x) = \frac{1}{5}(x + 2)^2(3x - 5)^2$

Using the Intermediate Value Theorem

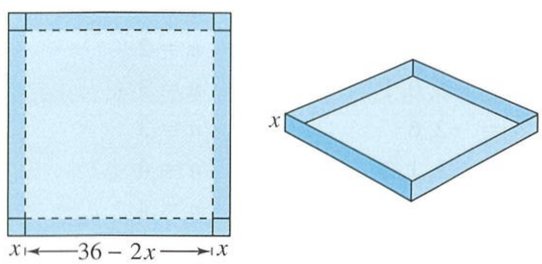
In Exercises 89–92, (a) use the Intermediate Value Theorem and the *table* feature of a graphing utility to find intervals one unit in length in which the polynomial function is guaranteed to have a zero. (b) Adjust the table to approximate the zeros of the function to the nearest thousandth.

89. $f(x) = x^3 - 3x^2 + 3$

90. $f(x) = 0.11x^3 - 2.07x^2 + 9.81x - 6.88$

91. $g(x) = 3x^4 + 4x^3 - 3$ **92.** $h(x) = x^4 - 10x^2 + 3$

93. Maximum Volume You construct an open box from a square piece of material, 36 inches on a side, by cutting equal squares with sides of length x from the corners and turning up the sides (see figure).



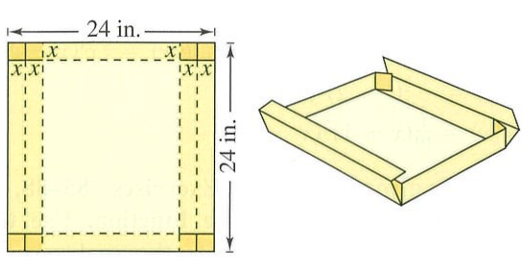
(a) Write a function V that represents the volume of the box.

(b) Determine the domain of the function V.

(c) Use a graphing utility to construct a table that shows the box heights x and the corresponding volumes $V(x)$. Use the table to estimate the dimensions that produce a maximum volume.

(d) Use the graphing utility to graph V and use the graph to estimate the value of x for which $V(x)$ is a maximum. Compare your result with that of part (c).

94. Maximum Volume You construct an open box with locking tabs from a square piece of material, 24 inches on a side, by cutting equal sections from the corners and folding along the dashed lines (see figure).



(a) Write a function V that represents the volume of the box.

(b) Determine the domain of the function V.

(c) Sketch a graph of the function and estimate the value of x for which $V(x)$ is a maximum.

95. Revenue The revenue R (in millions of dollars) for a software company from 2003 through 2016 can be modeled by

$$R = 6.212t^3 - 152.87t^2 + 990.2t - 414, \quad 3 \le t \le 16$$

where t represents the year, with $t = 3$ corresponding to 2003.

(a) Use a graphing utility to approximate any relative minima or maxima of the model over its domain.

(b) Use the graphing utility to approximate the intervals on which the revenue for the company is increasing and decreasing over its domain.

(c) Use the results of parts (a) and (b) to describe the company's revenue during this time period.

96. Revenue The revenue R (in millions of dollars) for a construction company from 2003 through 2010 can be modeled by

$$R = 0.1104t^4 - 5.152t^3 + 88.20t^2 - 654.8t + 1907, \quad 7 \le t \le 16$$

where t represents the year, with $t = 7$ corresponding to 2007.

(a) Use a graphing utility to approximate any relative minima or maxima of the model over its domain.

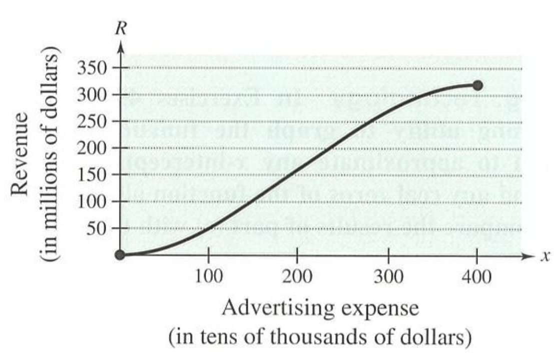
(b) Use the graphing utility to approximate the intervals on which the revenue for the company is increasing and decreasing over its domain.

(c) Use the results of parts (a) and (b) to describe the company's revenue during this time period.

97. Revenue The revenue R (in millions of dollars) for a beverage company is related to its advertising expense by the function

$$R = \frac{1}{100{,}000}(-x^3 + 600x^2), \quad 0 \le x \le 400$$

where x is the amount spent on advertising (in tens of thousands of dollars). Use the graph of this function to estimate the point on the graph at which the function is increasing most rapidly. This point is called the *point of diminishing returns* because any expense above this amount will yield less return per dollar invested in advertising.



98. Arboriculture The growth of a red oak tree is approximated by the function

$$G = -0.003t^3 + 0.137t^2 + 0.458t - 0.839, \quad 2 \le t \le 34$$

where G is the height of the tree (in feet) and t is its age (in years).

(a) Use a graphing utility to graph the function.

(b) Estimate the age of the tree when it is growing most rapidly. This point is called the *point of diminishing returns* because the increase in size will be less with each additional year.

(c) Using calculus, the point of diminishing returns can be found by finding the vertex of the parabola

$$y = -0.009t^2 + 0.274t + 0.458.$$

Find the vertex of this parabola.

(d) Compare your results from parts (b) and (c).

Exploration

True or False? **In Exercises 99–102, determine whether the statement is true or false. Justify your answer.**

99. If the graph of a polynomial function falls to the right, then its leading coefficient is negative.

100. A fifth-degree polynomial function can have five turning points in its graph.

101. It is possible for a polynomial with an even degree to have a range of $(-\infty, \infty)$.

102. If f is a polynomial function of x such that $f(2) = -6$ and $f(6) = 6$, then f has at most one real zero between $x = 2$ and $x = 6$.

103. Modeling Polynomials Sketch the graph of a fourth-degree polynomial function that has a zero of multiplicity 2 and a negative leading coefficient. Sketch the graph of another polynomial function with the same characteristics except that the leading coefficient is positive.

104. Modeling Polynomials Sketch the graph of a fifth-degree polynomial function that has a zero of multiplicity 2 and a negative leading coefficient. Sketch the graph of another polynomial function with the same characteristics except that the leading coefficient is positive.

105. Graphical Reasoning Sketch the graph of the function $f(x) = x^4$. Explain how the graph of each function g differs (if it does) from the graph of f. Determine whether g is even, odd, or neither.

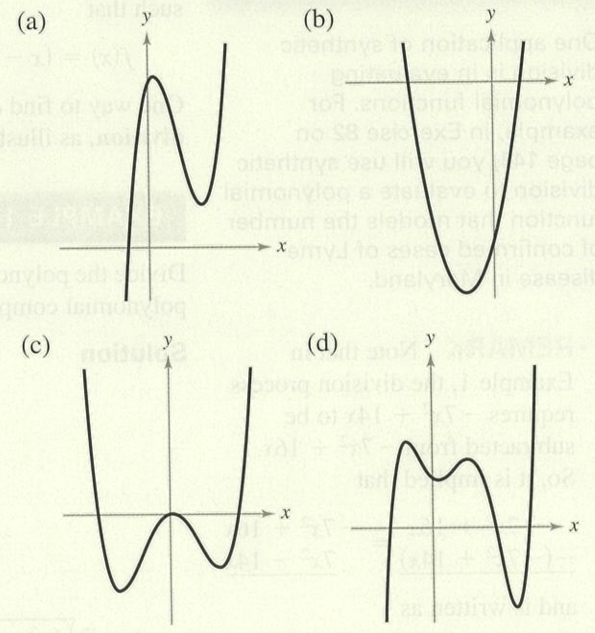
(a) $g(x) = f(x) + 2$ (b) $g(x) = f(x + 2)$

(c) $g(x) = f(-x)$ (d) $g(x) = -f(x)$

(e) $g(x) = f(\frac{1}{2}x)$ (f) $g(x) = \frac{1}{2}f(x)$

(g) $g(x) = f(x^{3/4})$ (h) $g(x) = (f \circ f)(x)$

106. HOW DO YOU SEE IT? For each graph, describe a polynomial function that could represent the graph. (Indicate the degree of the function and the sign of its leading coefficient.)



107. Think About It Use a graphing utility to graph the functions

$$y_1 = -\tfrac{1}{3}(x - 2)^5 + 1 \quad \text{and} \quad y_2 = \tfrac{3}{5}(x + 2)^5 - 3.$$

(a) Determine whether the graphs of y_1 and y_2 are increasing or decreasing. Explain.

(b) Will the graph of

$$g(x) = a(x - h)^5 + k$$

always be strictly increasing or strictly decreasing? If so, is this behavior determined by a, h, or k? Explain.

(c) Use a graphing utility to graph

$$f(x) = x^5 - 3x^2 + 2x + 1.$$

Use a graph and the result of part (b) to determine whether f can be written in the form $f(x) = a(x - h)^5 + k$. Explain.

Zigf | Dreamstime

2.3 Polynomial and Synthetic Division

One application of synthetic division is in evaluating polynomial functions. For example, in Exercise 82 on page 144, you will use synthetic division to evaluate a polynomial function that models the number of confirmed cases of Lyme disease in Maryland.

- **Use long division to divide polynomials by other polynomials.**
- **Use synthetic division to divide polynomials by binomials of the form $(x - k)$.**
- **Use the Remainder Theorem and the Factor Theorem.**

Long Division of Polynomials

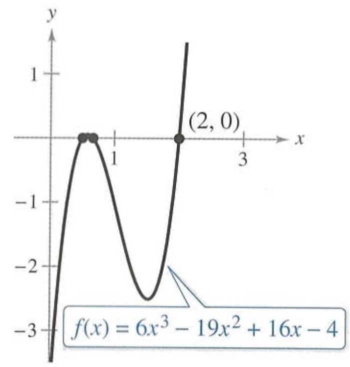
Consider the graph of

$$f(x) = 6x^3 - 19x^2 + 16x - 4$$

shown at the right. Notice that one of the zeros of f is $x = 2$. This means that $(x - 2)$ is a factor of $f(x)$, and there exists a second-degree polynomial $q(x)$ such that

$$f(x) = (x - 2) \cdot q(x).$$

One way to find $q(x)$ is to use **long division,** as illustrated in Example 1.



EXAMPLE 1 **Long Division of Polynomials**

Divide the polynomial $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

Solution

Think $\frac{6x^3}{x} = 6x^2$.

Think $\frac{-7x^2}{x} = -7x$.

Think $\frac{2x}{x} = 2$.

$$\begin{array}{r} 6x^2 - 7x + 2 \\ x - 2\overline{)\,6x^3 - 19x^2 + 16x - 4} \\ \underline{6x^3 - 12x^2} \qquad\qquad \\ -7x^2 + 16x \qquad \\ \underline{-7x^2 + 14x} \qquad \\ 2x - 4 \\ \underline{2x - 4} \\ 0 \end{array}$$

Multiply: $6x^2(x - 2)$.
Subtract and bring down $+ 16x$.
Multiply: $-7x(x - 2)$.
Subtract and bring down $- 4$.
Multiply: $2(x - 2)$.
Subtract.

From this division, you have shown that

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(6x^2 - 7x + 2)$$

and by factoring the quadratic $6x^2 - 7x + 2$, you have

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(2x - 1)(3x - 2).$$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Divide the polynomial $9x^3 + 36x^2 - 49x - 196$ by $x + 4$, and use the result to factor the polynomial completely.

REMARK Note that in Example 1, the division process requires $-7x^2 + 14x$ to be subtracted from $-7x^2 + 16x$. So, it is implied that

$$\begin{array}{r} -7x^2 + 16x \\ \underline{-(-7x^2 + 14x)} \end{array} = \begin{array}{r} -7x^2 + 16x \\ \underline{7x^2 - 14x} \end{array}$$

and is written as

$$\begin{array}{r} -7x^2 + 16x \\ \underline{-7x^2 + 14x} \\ 2x. \end{array}$$

REMARK Note that the factorization found in Example 1 agrees with the graph of f above. The three x-intercepts occur at $(2, 0)$, $\left(\frac{1}{2}, 0\right)$, and $\left(\frac{2}{3}, 0\right)$.

Dariusz Majgier/Shutterstock.com

In Example 1, $x - 2$ is a factor of the polynomial

$$6x^3 - 19x^2 + 16x - 4$$

and the long division process produces a remainder of zero. Often, long division will produce a nonzero remainder. For example, when you divide $x^2 + 3x + 5$ by $x + 1$, you obtain a remainder of 3.

$$\begin{array}{rrl} & x + 2 & \leftarrow \text{Quotient} \\ \text{Divisor} \rightarrow \quad x + 1 \,\big) & x^2 + 3x + 5 & \leftarrow \text{Dividend} \\ & \underline{x^2 + x} & \\ & 2x + 5 & \\ & \underline{2x + 2} & \\ & 3 & \leftarrow \text{Remainder} \end{array}$$

In fractional form, you can write this result as

$$\underbrace{\overbrace{\frac{x^3 + 3x + 5}{x + 1}}^{\text{Dividend}}}_{\text{Divisor}} = \overbrace{x + 2}^{\text{Quotient}} + \underbrace{\frac{\overset{\text{Remainder}}{3}}{x + 1}}_{\text{Divisor}}.$$

This implies that

$$x^2 + 3x + 5 = (x + 1)(x + 2) + 3 \qquad \text{Multiply each side by } (x + 1).$$

which illustrates a theorem called the **Division Algorithm.**

> **The Division Algorithm**
>
> If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that
>
> $$\underset{\text{Dividend}}{f(x)} = \underset{\text{Divisor}}{d(x)}\,\underset{\text{Quotient}}{q(x)} + \underset{\text{Remainder}}{r(x)}$$
>
> where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, then $d(x)$ *divides evenly* into $f(x)$.

Another way to write the Division Algorithm is

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

In the Division Algorithm, the rational expression $f(x)/d(x)$ is **improper** because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$. On the other hand, the rational expression $r(x)/d(x)$ is **proper** because the degree of $r(x)$ is less than the degree of $d(x)$.

If necessary, follow these steps before you apply the Division Algorithm.

1. Write the terms of the dividend and divisor in descending powers of the variable.
2. Insert placeholders with zero coefficients for missing powers of the variable.

Note how Examples 2 and 3 apply these steps.

EXAMPLE 2 Long Division of Polynomials

Divide $x^3 - 1$ by $x - 1$. Check the result.

Solution There is no x^2-term or x-term in the dividend $x^3 - 1$, so you need to rewrite the dividend as $x^3 + 0x^2 + 0x - 1$ before you apply the Division Algorithm.

$$\begin{array}{r l}
x^2 + x + 1 & \\
x - 1 \overline{) x^3 + 0x^2 + 0x - 1} & \\
\underline{x^3 - x^2} \qquad\qquad & \text{Multiply: } x^2(x - 1). \\
x^2 + 0x \qquad & \text{Subtract and bring down } 0x. \\
\underline{x^2 - x} \qquad & \text{Multiply: } x(x - 1). \\
x - 1 & \text{Subtract and bring down } -1. \\
\underline{x - 1} & \text{Multiply: } 1(x - 1). \\
0 & \text{Subtract.}
\end{array}$$

So, $x - 1$ divides evenly into $x^3 - 1$, and you can write

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1, \quad x \neq 1.$$

Check the result by multiplying.

$$\begin{aligned}(x - 1)(x^2 + x + 1) &= x^3 + x^2 + x - x^2 - x - 1 \\ &= x^3 - 1\end{aligned}$$

✓ *Checkpoint* *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Divide $x^3 - 2x^2 - 9$ by $x - 3$. Check the result.

EXAMPLE 3 Long Division of Polynomials

See LarsonPrecalculus.com for an interactive version of this type of example.

Divide $-5x^2 - 2 + 3x + 2x^4 + 4x^3$ by $2x - 3 + x^2$. Check the result.

Solution Write the terms of the dividend and divisor in descending powers of x.

$$\begin{array}{r l}
2x^2 \qquad\quad + 1 & \\
x^2 + 2x - 3 \overline{) 2x^4 + 4x^3 - 5x^2 + 3x - 2} & \\
\underline{2x^4 + 4x^3 - 6x^2} \qquad\qquad & \text{Multiply: } 2x^2(x^2 + 2x - 3). \\
x^2 + 3x - 2 & \text{Subtract and bring down } 3x - 2. \\
\underline{x^2 + 2x - 3} & \text{Multiply: } 1(x^2 + 2x - 3). \\
x + 1 & \text{Subtract.}
\end{array}$$

Note that the first subtraction eliminated two terms from the dividend. When this happens, the quotient skips a term. You can write the result as

$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}.$$

Check the result by multiplying.

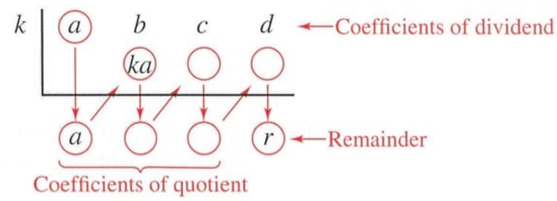
$$\begin{aligned}(x^2 + 2x - 3)(2x^2 + 1) + x + 1 &= 2x^4 + x^2 + 4x^3 + 2x - 6x^2 - 3 + x + 1 \\ &= 2x^4 + 4x^3 - 5x^2 + 3x - 2\end{aligned}$$

✓ *Checkpoint* *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Divide $-x^3 + 9x + 6x^4 - x^2 - 3$ by $1 + 3x$. Check the result.

Synthetic Division

For long division of polynomials by divisors of the form $x - k$, there is a shortcut called **synthetic division.** The pattern for synthetic division of a cubic polynomial is summarized below. (The pattern for higher-degree polynomials is similar.)

> **Synthetic Division (for a Cubic Polynomial)**
>
> To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use this pattern.
>
> 
>
>
> *Vertical pattern:* Add terms in columns.
> *Diagonal pattern:* Multiply results by k.

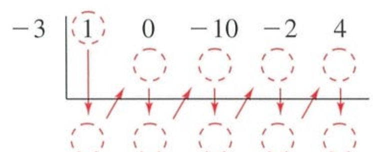
This algorithm for synthetic division works only for divisors of the form $x - k$. Remember that $x + k = x - (-k)$.

EXAMPLE 4 Using Synthetic Division

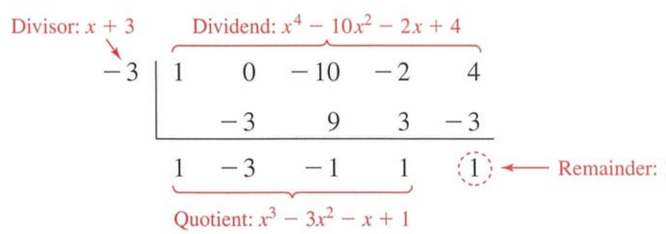
Use synthetic division to divide

$x^4 - 10x^2 - 2x + 4$ by $x + 3$.

Solution Begin by setting up an array. Include a zero for the missing x^3-term in the dividend.



Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3.



So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

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Use synthetic division to divide $5x^3 + 8x^2 - x + 6$ by $x + 2$.

The Remainder and Factor Theorems

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem.**

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is

$$r = f(k).$$

For a proof of the Remainder Theorem, see Proofs in Mathematics on page 193.

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$, as illustrated in Example 5.

EXAMPLE 5 Using the Remainder Theorem

Use the Remainder Theorem to evaluate

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

when $x = -2$. Check your answer.

Solution Using synthetic division gives the result below.

$$\begin{array}{r|rrrr} -2 & 3 & 8 & 5 & -7 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & -9 \end{array}$$

The remainder is $r = -9$, so

$$f(-2) = -9. \qquad r = f(k)$$

This means that $(-2, -9)$ is a point on the graph of f. Check this by substituting $x = -2$ in the original function.

Check

$$\begin{aligned} f(-2) &= 3(-2)^3 + 8(-2)^2 + 5(-2) - 7 \\ &= 3(-8) + 8(4) - 10 - 7 \\ &= -24 + 32 - 10 - 7 \\ &= -9 \end{aligned}$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Use the Remainder Theorem to find each function value given

$$f(x) = 4x^3 + 10x^2 - 3x - 8.$$

Check your answer.

a. $f(-1)$ **b.** $f(4)$

c. $f\left(\frac{1}{2}\right)$ **d.** $f(-3)$

TECHNOLOGY One way to evaluate a function with your graphing utility is to enter the function in the equation editor and use the *table* feature in *ask* mode. When you enter values in the X column of a table in *ask* mode, the corresponding function values are displayed in the function column.

Another important theorem is the **Factor Theorem,** stated below.

> **The Factor Theorem**
>
> A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

For a proof of the Factor Theorem, see Proofs in Mathematics on page 193.

Using the Factor Theorem, you can test whether a polynomial has $(x - k)$ as a factor by evaluating the polynomial at $x = k$. If the result is 0, then $(x - k)$ is a factor.

EXAMPLE 6 Factoring a Polynomial: Repeated Division

Show that $(x - 2)$ and $(x + 3)$ are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

Algebraic Solution

Using synthetic division with the factor $(x - 2)$ gives the result below.

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & 0 \end{array}$$

0 remainder, so $f(2) = 0$ and $(x - 2)$ is a factor.

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & 0 \end{array}$$

$2x^2 + 5x + 3$

0 remainder, so $f(-3) = 0$ and $(x + 3)$ is a factor.

The resulting quadratic expression factors as

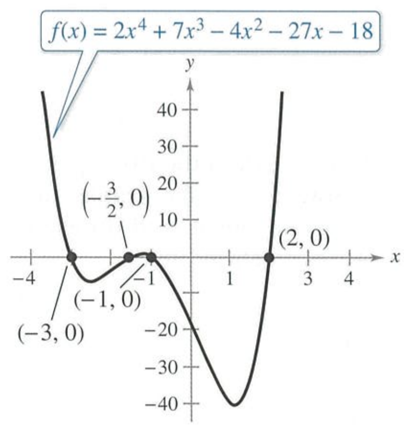
$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

so the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

Graphical Solution

The graph of $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$ has four x-intercepts (see figure). These occur at $x = -3$, $x = -\frac{3}{2}$, $x = -1$, and $x = 2$. (Check this algebraically.) This implies that $(x + 3)$, $\left(x + \frac{3}{2}\right)$, $(x + 1)$, and $(x - 2)$ are factors of $f(x)$. $\left[\text{Note that } \left(x + \frac{3}{2}\right) \text{ and } (2x + 3) \text{ are equivalent factors because they both yield the same zero, } x = -\frac{3}{2}.\right]$



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Show that $(x + 3)$ is a factor of $f(x) = x^3 - 19x - 30$. Then find the remaining factors of $f(x)$.

> **Summarize (Section 2.3)**
>
> 1. Explain how to use long division to divide two polynomials *(pages 136 and 137)*. For examples of long division of polynomials, see Examples 1–3.
> 2. Describe the algorithm for synthetic division *(page 139)*. For an example of synthetic division, see Example 4.
> 3. State the Remainder Theorem and the Factor Theorem *(pages 140 and 141)*. For an example of using the Remainder Theorem, see Example 5. For an example of using the Factor Theorem, see Example 6.

2.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary

1. Two forms of the Division Algorithm are shown below. Identify and label each term or function.

$$f(x) = d(x)q(x) + r(x) \qquad \frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

In Exercises 2–6, fill in the blanks.

2. In the Division Algorithm, the rational expression $r(x)/d(x)$ is ________ because the degree of $r(x)$ is less than the degree of $d(x)$.
3. In the Division Algorithm, the rational expression $f(x)/d(x)$ is ________ because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$.
4. A shortcut for long division of polynomials is ________ ________, in which the divisor must be of the form $x - k$.
5. The ________ Theorem states that a polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.
6. The ________ Theorem states that if a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

Skills and Applications

Using the Division Algorithm **In Exercises 7 and 8, use long division to verify that $y_1 = y_2$.**

7. $y_1 = \dfrac{x^2}{x+2}, \quad y_2 = x - 2 + \dfrac{4}{x+2}$
8. $y_1 = \dfrac{x^3 - 3x^2 + 4x - 1}{x+3}, \quad y_2 = x^2 - 6x + 22 - \dfrac{67}{x+3}$

Using Technology **In Exercises 9 and 10, (a) use a graphing utility to graph the two equations in the same viewing window, (b) use the graphs to verify that the expressions are equivalent, and (c) use long division to verify the results algebraically.**

9. $y_1 = \dfrac{x^2 + 2x - 1}{x+3}, \quad y_2 = x - 1 + \dfrac{2}{x+3}$
10. $y_1 = \dfrac{x^4 + x^2 - 1}{x^2+1}, \quad y_2 = x^2 - \dfrac{1}{x^2+1}$

Long Division of Polynomials **In Exercises 11–24, use long division to divide.**

11. $(2x^2 + 10x + 12) \div (x + 3)$
12. $(5x^2 - 17x - 12) \div (x - 4)$
13. $(4x^3 - 7x^2 - 11x + 5) \div (4x + 5)$
14. $(6x^3 - 16x^2 + 17x - 6) \div (3x - 2)$
15. $(x^4 + 5x^3 + 6x^2 - x - 2) \div (x + 2)$
16. $(x^3 + 4x^2 - 3x - 12) \div (x - 3)$
17. $(6x + 5) \div (x + 1)$
18. $(9x - 4) \div (3x + 2)$
19. $(x^3 - 9) \div (x^2 + 1)$
20. $(x^5 + 7) \div (x^4 - 1)$
21. $(3x + 2x^3 - 9 - 8x^2) \div (x^2 + 1)$
22. $(5x^3 - 16 - 20x + x^4) \div (x^2 - x - 3)$
23. $\dfrac{x^4}{(x-1)^3}$
24. $\dfrac{2x^3 - 4x^2 - 15x + 5}{(x-1)^2}$

Using Synthetic Division **In Exercises 25–44, use synthetic division to divide.**

25. $(2x^3 - 10x^2 + 14x - 24) \div (x - 4)$
26. $(5x^3 + 18x^2 + 7x - 6) \div (x + 3)$
27. $(6x^3 + 7x^2 - x + 26) \div (x - 3)$
28. $(2x^3 + 12x^2 + 14x - 3) \div (x + 4)$
29. $(4x^3 - 9x + 8x^2 - 18) \div (x + 2)$
30. $(9x^3 - 16x - 18x^2 + 32) \div (x - 2)$
31. $(-x^3 + 75x - 250) \div (x + 10)$
32. $(3x^3 - 16x^2 - 72) \div (x - 6)$
33. $(x^3 - 3x^2 + 5) \div (x - 4)$
34. $(5x^3 + 6x + 8) \div (x + 2)$
35. $\dfrac{10x^4 - 50x^3 - 800}{x - 6}$
36. $\dfrac{x^5 - 13x^4 - 120x + 80}{x + 3}$
37. $\dfrac{x^3 + 512}{x + 8}$
38. $\dfrac{x^3 - 729}{x - 9}$
39. $\dfrac{-3x^4}{x - 2}$
40. $\dfrac{-2x^5}{x + 2}$
41. $\dfrac{180x - x^4}{x - 6}$
42. $\dfrac{5 - 3x + 2x^2 - x^3}{x + 1}$
43. $\dfrac{4x^3 + 16x^2 - 23x - 15}{x + \frac{1}{2}}$
44. $\dfrac{3x^3 - 4x^2 + 5}{x - \frac{3}{2}}$

Using the Remainder Theorem **In Exercises 45–50, write the function in the form $f(x) = (x - k)q(x) + r$ for the given value of k, and demonstrate that $f(k) = r$.**

45. $f(x) = x^3 - x^2 - 10x + 7, \quad k = 3$
46. $f(x) = x^3 - 4x^2 - 10x + 8, \quad k = -2$
47. $f(x) = 15x^4 + 10x^3 - 6x^2 + 14, \quad k = -\frac{2}{3}$
48. $f(x) = 10x^3 - 22x^2 - 3x + 4, \quad k = \frac{1}{5}$
49. $f(x) = -4x^3 + 6x^2 + 12x + 4, \quad k = 1 - \sqrt{3}$
50. $f(x) = -3x^3 + 8x^2 + 10x - 8, \quad k = 2 + \sqrt{2}$

Using the Remainder Theorem **In Exercises 51–54, use the Remainder Theorem and synthetic division to find each function value. Verify your answers using another method.**

51. $f(x) = 2x^3 - 7x + 3$
    (a) $f(1)$ (b) $f(-2)$ (c) $f(3)$ (d) $f(2)$
52. $g(x) = 2x^6 + 3x^4 - x^2 + 3$
    (a) $g(2)$ (b) $g(1)$ (c) $g(3)$ (d) $g(-1)$
53. $h(x) = x^3 - 5x^2 - 7x + 4$
    (a) $h(3)$ (b) $h(\frac{1}{2})$ (c) $h(-2)$ (d) $h(-5)$
54. $f(x) = 4x^4 - 16x^3 + 7x^2 + 20$
    (a) $f(1)$ (b) $f(-2)$ (c) $f(5)$ (d) $f(-10)$

Using the Factor Theorem **In Exercises 55–62, use synthetic division to show that x is a solution of the third-degree polynomial equation, and use the result to factor the polynomial completely. List all real solutions of the equation.**

55. $x^3 + 6x^2 + 11x + 6 = 0, \quad x = -3$
56. $x^3 - 52x - 96 = 0, \quad x = -6$
57. $2x^3 - 15x^2 + 27x - 10 = 0, \quad x = \frac{1}{2}$
58. $48x^3 - 80x^2 + 41x - 6 = 0, \quad x = \frac{2}{3}$
59. $x^3 + 2x^2 - 3x - 6 = 0, \quad x = \sqrt{3}$
60. $x^3 + 2x^2 - 2x - 4 = 0, \quad x = \sqrt{2}$
61. $x^3 - 3x^2 + 2 = 0, \quad x = 1 + \sqrt{3}$
62. $x^3 - x^2 - 13x - 3 = 0, \quad x = 2 - \sqrt{5}$

Factoring a Polynomial **In Exercises 63–70, (a) verify the given factors of $f(x)$, (b) find the remaining factor(s) of $f(x)$, (c) use your results to write the complete factorization of $f(x)$, (d) list all real zeros of f, and (e) confirm your results by using a graphing utility to graph the function.**

	Function	Factors
63.	$f(x) = 2x^3 + x^2 - 5x + 2$	$(x + 2), (x - 1)$
64.	$f(x) = 3x^3 - x^2 - 8x - 4$	$(x + 1), (x - 2)$
65.	$f(x) = x^4 - 8x^3 + 9x^2 + 38x - 40$	$(x - 5), (x + 2)$
66.	$f(x) = 8x^4 - 14x^3 - 71x^2 - 10x + 24$	$(x + 2), (x - 4)$
67.	$f(x) = 6x^3 + 41x^2 - 9x - 14$	$(2x + 1), (3x - 2)$
68.	$f(x) = 10x^3 - 11x^2 - 72x + 45$	$(2x + 5), (5x - 3)$
69.	$f(x) = 2x^3 - x^2 - 10x + 5$	$(2x - 1), (x + \sqrt{5})$
70.	$f(x) = x^3 + 3x^2 - 48x - 144$	$(x + 4\sqrt{3}), (x + 3)$

Approximating Zeros **In Exercises 71–76, (a) use the *zero* or *root* feature of a graphing utility to approximate the zeros of the function accurate to three decimal places, (b) determine the exact value of one of the zeros, and (c) use synthetic division to verify your result from part (b), and then factor the polynomial completely.**

71. $f(x) = x^3 - 2x^2 - 5x + 10$
72. $g(x) = x^3 + 3x^2 - 2x - 6$
73. $h(t) = t^3 - 2t^2 - 7t + 2$
74. $f(s) = s^3 - 12s^2 + 40s - 24$
75. $h(x) = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$
76. $g(x) = 6x^4 - 11x^3 - 51x^2 + 99x - 27$

Simplifying Rational Expressions **In Exercises 77–80, simplify the rational expression by using long division or synthetic division.**

77. $\dfrac{x^3 + x^2 - 64x - 64}{x + 8}$
78. $\dfrac{4x^3 - 8x^2 + x + 3}{2x - 3}$
79. $\dfrac{x^4 + 6x^3 + 11x^2 + 6x}{x^2 + 3x + 2}$
80. $\dfrac{x^4 + 9x^3 - 5x^2 - 36x + 4}{x^2 - 4}$

81. **Profit** A company that produces calculators estimates that the profit P (in dollars) from selling a specific model of calculator is given by

$$P = -152x^3 + 7545x^2 - 169{,}625, \quad 0 \le x \le 45$$

where x is the advertising expense (in tens of thousands of dollars). For this model of calculator, an advertising expense of \$400,000 ($x = 40$) results in a profit of \$2,174,375.

(a) Use a graphing utility to graph the profit function.
(b) Use the graph from part (a) to estimate another amount the company can spend on advertising that results in the same profit.
(c) Use synthetic division to confirm the result of part (b) algebraically.

82. Lyme Disease The numbers N of confirmed cases of Lyme disease in Maryland from 2007 through 2014 are shown in the table, where t represents the year, with $t = 7$ corresponding to 2007. *(Source: Centers for Disease Control and Prevention)*

DATA

Spreadsheet at LarsonPrecalculus.com

Year, t	Number, N
7	2576
8	1746
9	1466
10	1163
11	938
12	1113
13	801
14	957

(a) Use a graphing utility to create a scatter plot of the data.

(b) Use the *regression* feature of the graphing utility to find a *quartic* model for the data. (A quartic model has the form $at^4 + bt^3 + ct^2 + dt + e$, where a, b, c, d, and e are constant and t is variable.) Graph the model in the same viewing window as the scatter plot.

(c) Use the model to create a table of estimated values of N. Compare the model with the original data.

(d) Use synthetic division to confirm algebraically your estimated value for the year 2014.

Exploration

True or False? **In Exercises 83–86, determine whether the statement is true or false. Justify your answer.**

83. If $(7x + 4)$ is a factor of some polynomial function $f(x)$, then $\frac{4}{7}$ is a zero of f.

84. $(2x - 1)$ is a factor of the polynomial

$$6x^6 + x^5 - 92x^4 + 45x^3 + 184x^2 + 4x - 48.$$

85. The rational expression $\dfrac{x^3 + 2x^2 - 7x + 4}{x^2 - 4x - 12}$ is improper.

86. The equation

$$\frac{x^3 - 3x^2 + 4}{x + 1} = x^2 - 4x + 4$$

is true for all values of x.

Think About It **In Exercises 87 and 88, perform the division. Assume that n is a positive integer.**

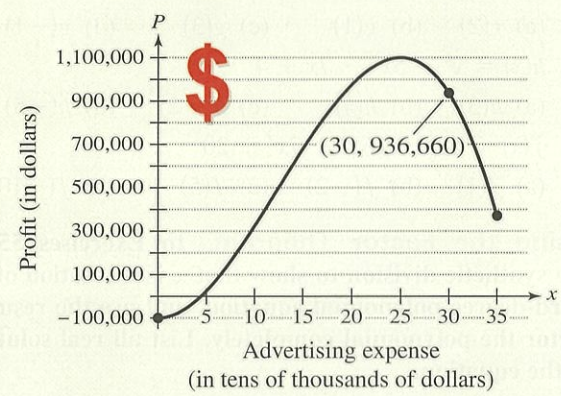
87. $\dfrac{x^{3n} + 9x^{2n} + 27x^n + 27}{x^n + 3}$ **88.** $\dfrac{x^{3n} - 3x^{2n} + 5x^n - 6}{x^n - 2}$

89. Error Analysis Describe the error.

Use synthetic division to find the remainder when $x^2 + 3x - 5$ is divided by $x + 1$.

$$\begin{array}{r|rrr} 1 & 1 & 3 & -5 \\ & & 1 & 4 \\ \hline & 1 & 4 & -1 \end{array} \quad \leftarrow \text{Remainder: } -1 \quad \times$$

90. HOW DO YOU SEE IT? The graph below shows a company's estimated profits for different advertising expenses. The company's actual profit was \$936,660 for an advertising expense of \$300,000.



(a) From the graph, it appears that the company could have obtained the same profit for a lesser advertising expense. Use the graph to estimate this expense.

(b) The company's model is

$$P = -140.75x^3 + 5348.3x^2 - 76{,}560, \quad 0 \le x \le 35$$

where P is the profit (in dollars) and x is the advertising expense (in tens of thousands of dollars). Explain how you could verify the lesser expense from part (a) algebraically.

Exploration **In Exercises 91 and 92, find the constant c such that the denominator will divide evenly into the numerator.**

91. $\dfrac{x^3 + 4x^2 - 3x + c}{x - 5}$ **92.** $\dfrac{x^5 - 2x^2 + x + c}{x + 2}$

93. Think About It Find the value of k such that $x - 4$ is a factor of $x^3 - kx^2 + 2kx - 8$.