10.2 Introduction to Conics: Parabolas

Parabolas have many real-life applications and are often used to model and solve engineering problems. For example, in Exercise 72 on page 706, you will use a parabola to model the cables of the Golden Gate Bridge.

- **Recognize a conic as the intersection of a plane and a double-napped cone.**
- **Write equations of parabolas in standard form.**
- **Use the reflective property of parabolas to write equations of tangent lines.**

Conics

The earliest basic descriptions of conic sections took place during the classical Greek period, 500 to 336 B.C. This early Greek study was largely concerned with the geometric properties of conics. It was not until the early 17th century that the broad applicability of conics became apparent and played a prominent role in the early development of calculus.

A **conic section** (or simply **conic**) is the intersection of a plane and a double-napped cone. Notice in Figure 10.7 that in the formation of the four basic conics, the intersecting plane does not pass through the vertex of the cone. When the plane does pass through the vertex, the resulting figure is a **degenerate conic,** as shown in Figure 10.8.

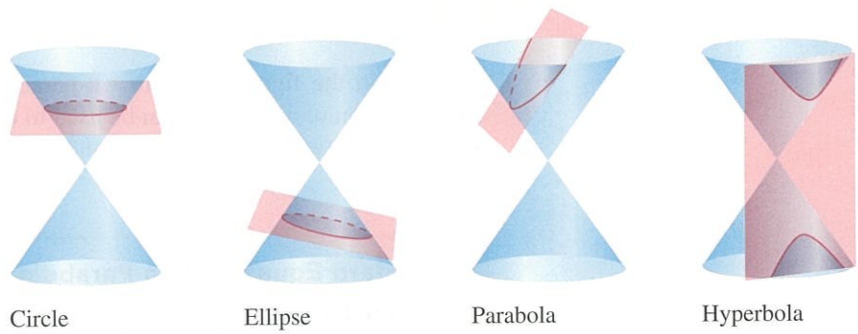

Figure 10.7 Basic Conics

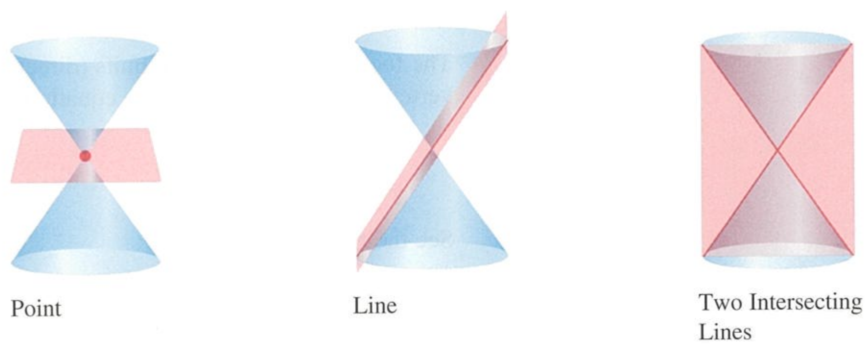

Figure 10.8 Degenerate Conics

There are several ways to approach the study of conics. You could begin by defining conics in terms of the intersections of planes and cones, as the Greeks did, or you could define them algebraically, in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

However, you will study a third approach, in which each of the conics is defined as a *locus* (collection) of points satisfying a given geometric property. For example, in Section 1.2, you saw how the definition of a circle as *the collection of all points* (x, y) *that are equidistant from a fixed point* (h, k) led to the standard form of the equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2. \qquad \text{Equation of a circle}$$

Recall that the center of a circle is at (h, k) and that the radius of the circle is r.

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Parabolas

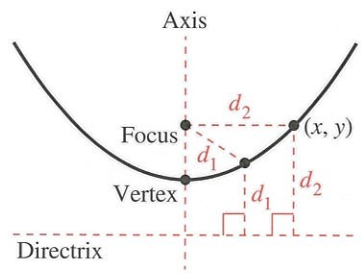
In Section 2.1, you learned that the graph of the quadratic function

$$f(x) = ax^2 + bx + c$$

is a parabola that opens upward or downward. The definition of a parabola below is more general in the sense that it is independent of the orientation of the parabola.

Definition of a Parabola

A **parabola** is the set of all points (x, y) in a plane that are equidistant from a fixed line, the **directrix,** and a fixed point, the **focus,** not on the line. (See figure.) The **vertex** is the midpoint between the focus and the directrix. The **axis** of the parabola is the line passing through the focus and the vertex.



Note in the figure above that a parabola is symmetric with respect to its axis. The definition of a parabola can be used to derive the **standard form of the equation of a parabola** with vertex at (h, k) and directrix parallel to the x-axis or to the y-axis, stated below.

Standard Equation of a Parabola

The **standard form of the equation of a parabola** with vertex at (h, k) is

$(x - h)^2 = 4p(y - k), \quad p \neq 0$ Vertical axis; directrix: $y = k - p$

$(y - k)^2 = 4p(x - h), \quad p \neq 0.$ Horizontal axis; directrix: $x = h - p$

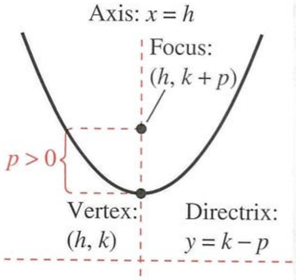
The focus lies on the axis p units (directed distance) from the vertex. If the vertex is at the origin, then the equation takes one of two forms.

$x^2 = 4py$ Vertical axis

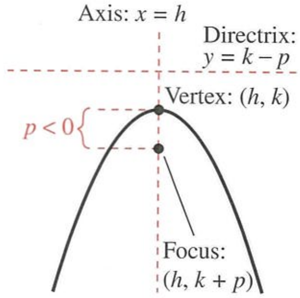
$y^2 = 4px$ Horizontal axis

See the figures below.

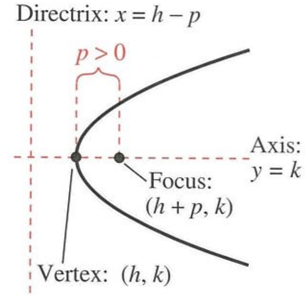
For a proof of the standard form of the equation of a parabola, see Proofs in Mathematics on page 773.



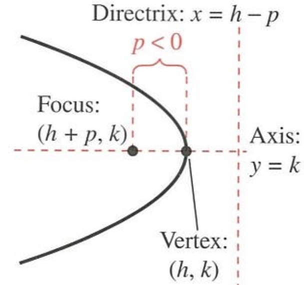
$(x - h)^2 = 4p(y - k)$
Vertical axis: $p > 0$



$(x - h)^2 = 4p(y - k)$
Vertical axis: $p < 0$



$(y - k)^2 = 4p(x - h)$
Horizontal axis: $p > 0$



$(y - k)^2 = 4p(x - h)$
Horizontal axis: $p < 0$

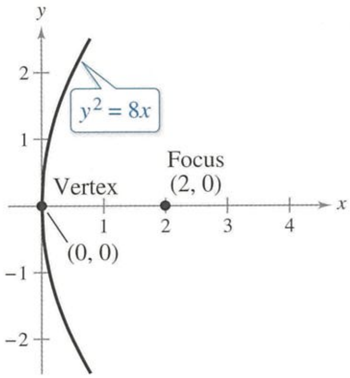


Figure 10.9

EXAMPLE 1 Finding the Standard Equation of a Parabola

Find the standard form of the equation of the parabola with vertex at the origin and focus $(2, 0)$.

Solution The axis of the parabola is horizontal, passing through $(0, 0)$ and $(2, 0)$, as shown in Figure 10.9. The equation is of the form $y^2 = 4px$, where $p = 2$. So, the standard form of the equation is $y^2 = 8x$. You can use a graphing utility to confirm this equation. Let $y_1 = \sqrt{8x}$ to graph the upper portion of the parabola and let $y_2 = -\sqrt{8x}$ to graph the lower portion of the parabola.

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Find the standard form of the equation of the parabola with vertex at the origin and focus $\left(0, \frac{3}{8}\right)$.

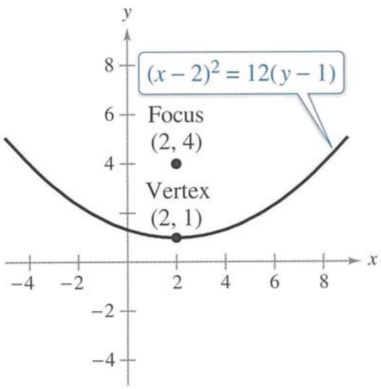


Figure 10.10

EXAMPLE 2 Finding the Standard Equation of a Parabola

See LarsonPrecalculus.com for an interactive version of this type of example.

Find the standard form of the equation of the parabola with vertex $(2, 1)$ and focus $(2, 4)$.

Solution The axis of the parabola is vertical, passing through $(2, 1)$ and $(2, 4)$. The equation is of the form

$$(x - h)^2 = 4p(y - k)$$

where $h = 2$, $k = 1$, and $p = 4 - 1 = 3$. So, the standard form of the equation is

$$(x - 2)^2 = 12(y - 1).$$

Figure 10.10 shows the graph of this parabola.

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Find the standard form of the equation of the parabola with vertex $(2, -3)$ and focus $(4, -3)$.

ALGEBRA HELP The technique of completing the square is used to write the equation in Example 3 in standard form. To review completing the square, see Appendix A.5.

EXAMPLE 3 Finding the Focus of a Parabola

Find the focus of the parabola $y = -\frac{1}{2}x^2 - x + \frac{1}{2}$.

Solution Convert to standard form by completing the square.

$y = -\frac{1}{2}x^2 - x + \frac{1}{2}$	Write original equation.
$-2y = x^2 + 2x - 1$	Multiply each side by -2.
$1 - 2y = x^2 + 2x$	Add 1 to each side.
$1 + 1 - 2y = x^2 + 2x + 1$	Complete the square.
$2 - 2y = x^2 + 2x + 1$	Combine like terms.
$-2(y - 1) = (x + 1)^2$	Write in standard form.

Comparing this equation with

$$(x - h)^2 = 4p(y - k)$$

shows that $h = -1$, $k = 1$, and $p = -\frac{1}{2}$. The parabola opens downward, as shown in Figure 10.11, because p is negative. So, the focus of the parabola is $(h, k + p) = \left(-1, \frac{1}{2}\right)$.

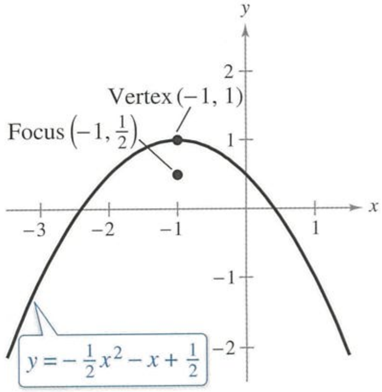


Figure 10.11

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

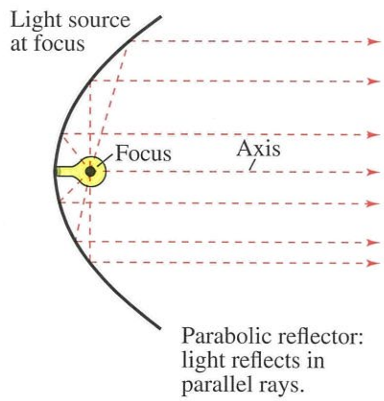
Find the focus of the parabola $x = \frac{1}{4}y^2 + \frac{3}{2}y + \frac{13}{4}$.

One important application of parabolas is in astronomy. Radio telescopes use parabolic dishes to collect radio waves from space.

The Reflective Property of Parabolas

A line segment that passes through the focus of a parabola and has endpoints on the parabola is a **focal chord.** The focal chord perpendicular to the axis of the parabola is called the **latus rectum.**

Parabolas occur in a wide variety of applications. For example, a parabolic reflector can be formed by revolving a parabola about its axis. The resulting surface has the property that all incoming rays parallel to the axis reflect through the focus of the parabola. This is the principle behind the construction of the parabolic mirrors used in reflecting telescopes. Conversely, the light rays emanating from the focus of a parabolic reflector used in a flashlight are all parallel to one another, as shown in the figure below.

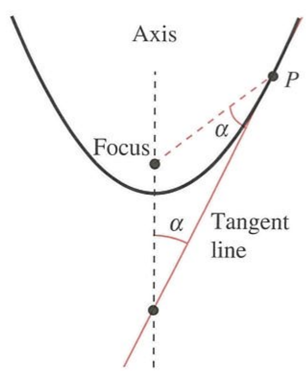


A line is **tangent** to a parabola at a point on the parabola when the line intersects, but does not cross, the parabola at the point. Tangent lines to parabolas have special properties related to the use of parabolas in constructing reflective surfaces.

Reflective Property of a Parabola

The tangent line to a parabola at a point P makes equal angles with the following two lines (see figure below).

1. The line passing through P and the focus
2. The axis of the parabola



Example 4 shows how to find an equation of a tangent line to a parabola at a given point. Finding slopes and equations of tangent lines are important topics in calculus. If you take a calculus course, you will study techniques for finding slopes and equations of tangent lines to parabolas and other curves.

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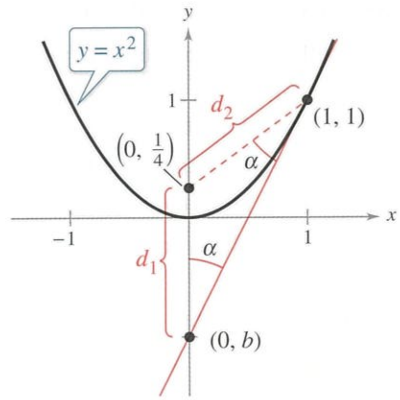
EXAMPLE 4 Finding the Tangent Line at a Point on a Parabola

Find an equation of the tangent line to the parabola $y = x^2$ at the point $(1, 1)$.

Solution For this parabola, the vertex is at the origin, the axis is vertical, and $p = \frac{1}{4}$, so the focus is $\left(0, \frac{1}{4}\right)$, as shown in the figure below.



To find the y-intercept $(0, b)$ of the tangent line, equate the lengths of the two sides of the isosceles triangle shown in the figure:

$$d_1 = \frac{1}{4} - b$$

and

$$d_2 = \sqrt{(1-0)^2 + \left(1 - \frac{1}{4}\right)^2} = \frac{5}{4}.$$

Note that $d_1 = \frac{1}{4} - b$ rather than $b - \frac{1}{4}$. The order of subtraction for the distance is important because the distance must be positive. Setting $d_1 = d_2$ produces

$$\frac{1}{4} - b = \frac{5}{4}$$

$$b = -1.$$

So, the slope of the tangent line is

$$m = \frac{1 - (-1)}{1 - 0} = 2$$

and the equation of the tangent line in slope-intercept form is

$$y = 2x - 1.$$

TECHNOLOGY Use a graphing utility to confirm the result of Example 4. Graph

$$y_1 = x^2 \quad \text{and} \quad y_2 = 2x - 1$$

in the same viewing window and verify that the line touches the parabola at the point $(1, 1)$.

ALGEBRA HELP To review techniques for writing linear equations, see Section 1.3.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Find an equation of the tangent line to the parabola $y = 3x^2$ at the point $(1, 3)$.

Summarize (Section 10.2)

1. List the four basic conic sections and the degenerate conics. Use sketches to show how to form each basic conic section and degenerate conic from the intersection of a plane and a double-napped cone *(page 699)*.
2. State the definition of a parabola and the standard form of the equation of a parabola *(page 700)*. For examples involving writing equations of parabolas in standard form, see Examples 1–3.
3. State the reflective property of a parabola *(page 702)*. For an example of using this property to write an equation of a tangent line, see Example 4.

10.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

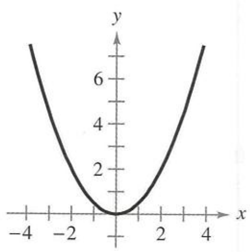
Vocabulary: Fill in the blanks.

1. A ________ is the intersection of a plane and a double-napped cone.
2. When a plane passes through the vertex of a double-napped cone, the intersection is a ________ ________.
3. A ________ of points is a collection of points satisfying a given geometric property.
4. A ________ is the set of all points (x, y) in a plane that are equidistant from a fixed line, called the ________, and a fixed point, called the ________, not on the line.
5. The line that passes through the focus and the vertex of a parabola is the ________ of the parabola.
6. The ________ of a parabola is the midpoint between the focus and the directrix.
7. A line segment that passes through the focus of a parabola and has endpoints on the parabola is a ________ ________.
8. A line is ________ to a parabola at a point on the parabola when the line intersects, but does not cross, the parabola at the point.

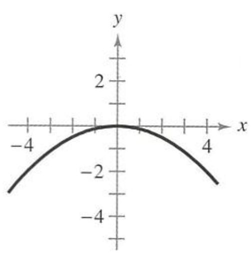
Skills and Applications

Matching **In Exercises 9–12, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

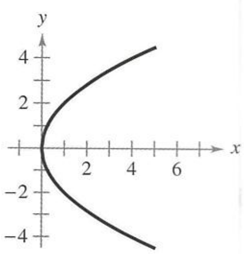
(a)



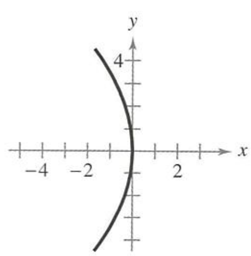
(b)



(c)



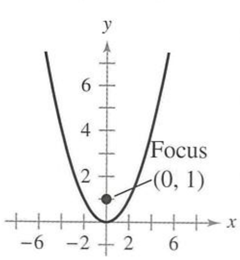
(d)



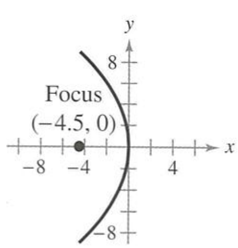
9. $y^2 = 4x$
10. $x^2 = 2y$
11. $x^2 = -8y$
12. $y^2 = -12x$

Finding the Standard Equation of a Parabola **In Exercises 13–26, find the standard form of the equation of the parabola with the given characteristic(s) and vertex at the origin.**

13.



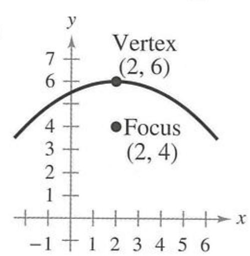
14.



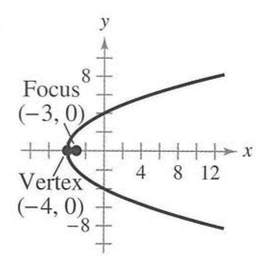
15. Focus: $\left(0, \frac{1}{2}\right)$
16. Focus: $\left(\frac{3}{2}, 0\right)$
17. Focus: $(-2, 0)$
18. Focus: $(0, -1)$
19. Directrix: $y = 2$
20. Directrix: $y = -4$
21. Directrix: $x = -1$
22. Directrix: $x = 3$
23. Vertical axis; passes through the point $(4, 6)$
24. Vertical axis; passes through the point $(-3, -3)$
25. Horizontal axis; passes through the point $(-2, 5)$
26. Horizontal axis; passes through the point $(3, -2)$

Finding the Standard Equation of a Parabola **In Exercises 27–36, find the standard form of the equation of the parabola with the given characteristics.**

27.



28.



29. Vertex: $(6, 3)$; focus: $(4, 3)$
30. Vertex: $(1, -8)$; focus: $(3, -8)$
31. Vertex: $(0, 2)$; directrix: $y = 4$
32. Vertex: $(1, 2)$; directrix: $y = -1$
33. Focus: $(2, 2)$; directrix: $x = -2$
34. Focus: $(0, 0)$; directrix: $y = 8$
35. Vertex: $(3, -3)$; vertical axis; passes through the point $(0, 0)$
36. Vertex: $(-1, 6)$; horizontal axis; passes through the point $(-9, 2)$

Finding the Vertex, Focus, and Directrix of a Parabola **In Exercises 37–50, find the vertex, focus, and directrix of the parabola. Then sketch the parabola.**

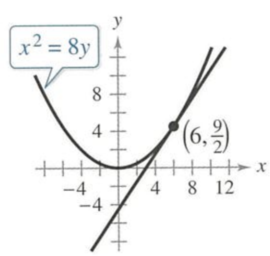
37. $y = \frac{1}{2}x^2$
38. $y = -4x^2$
39. $y^2 = -6x$
40. $y^2 = 3x$
41. $x^2 + 12y = 0$
42. $x + y^2 = 0$
43. $(x - 1)^2 + 8(y + 2) = 0$
44. $(x + 5) + (y - 1)^2 = 0$
45. $(y + 7)^2 = 4\left(x - \frac{3}{2}\right)$
46. $\left(x + \frac{1}{2}\right)^2 = 4(y - 1)$
47. $y = \frac{1}{4}(x^2 - 2x + 5)$
48. $x = \frac{1}{4}(y^2 + 2y + 33)$
49. $y^2 + 6y + 8x + 25 = 0$
50. $x^2 - 4x - 4y = 0$

Finding the Vertex, Focus, and Directrix of a Parabola **In Exercises 51–54, find the vertex, focus, and directrix of the parabola. Use a graphing utility to graph the parabola.**

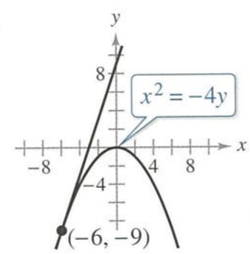
51. $x^2 + 4x - 6y = -10$
52. $x^2 - 2x + 8y = -9$
53. $y^2 + x + y = 0$
54. $y^2 - 4x - 4 = 0$

Finding the Tangent Line at a Point on a Parabola **In Exercises 55–60, find an equation of the tangent line to the parabola at the given point.**

55.



56.



57. $x^2 = 2y$, $(4, 8)$
58. $x^2 = 2y$, $\left(-3, \frac{9}{2}\right)$
59. $y = -2x^2$, $(-1, -2)$
60. $y = -2x^2$, $(2, -8)$

61. **Flashlight** The light bulb in a flashlight is at the focus of the parabolic reflector, 1.5 centimeters from the vertex of the reflector (see figure). Write an equation for a cross section of the flashlight's reflector with its focus on the positive x-axis and its vertex at the origin.

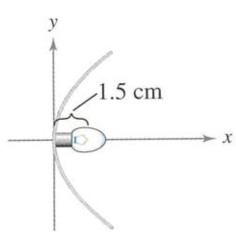


Figure for 61

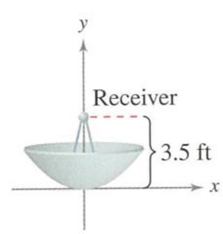
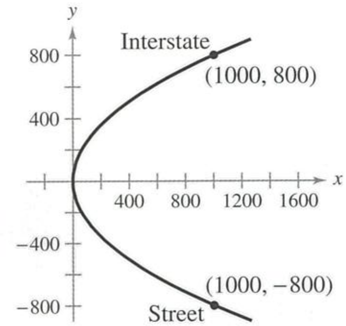


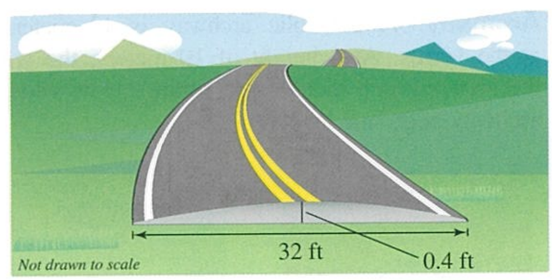
Figure for 62

62. **Satellite Dish** The receiver of a parabolic satellite dish is at the focus of the parabola (see figure). Write an equation for a cross section of the satellite dish.

63. **Highway Design** Highway engineers use a parabolic curve to design an entrance ramp from a straight street to an interstate highway (see figure). Write an equation of the parabola.



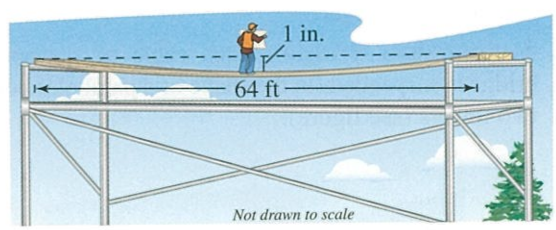
64. **Road Design** Roads are often designed with parabolic surfaces to allow rain to drain off. A particular road is 32 feet wide and 0.4 foot higher in the center than it is on the sides (see figure).



(a) Write an equation of the parabola with its vertex at the origin that models the road surface.

(b) How far from the center of the road is the road surface 0.1 foot lower than the center?

65. **Beam Deflection** A simply supported beam is 64 feet long and has a load at the center (see figure). The deflection of the beam at its center is 1 inch. The shape of the deflected beam is parabolic.



(a) Write an equation of the parabola with its vertex at the origin that models the shape of the beam.

(b) How far from the center of the beam is the deflection $\frac{1}{2}$ inch?

66. **Beam Deflection** Repeat Exercise 65 when the length of the beam is 36 feet and the deflection of the beam at its center is 2 inches.

67. Fluid Flow Water is flowing from a horizontal pipe 48 feet above the ground. The falling stream of water has the shape of a parabola whose vertex $(0, 48)$ is at the end of the pipe (see figure). The stream of water strikes the ocean at the point $(10\sqrt{3}, 0)$. Write an equation for the path of the water.

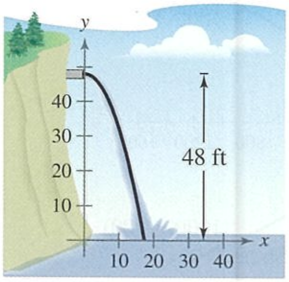


Figure for 67

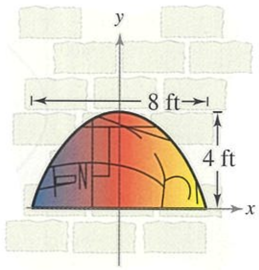


Figure for 68

68. Window Design A church window is bounded above by a parabola (see figure). Write an equation of the parabola.

69. Archway A parabolic archway is 12 meters high at the vertex. At a height of 10 meters, the width of the archway is 8 meters (see figure). How wide is the archway at ground level?

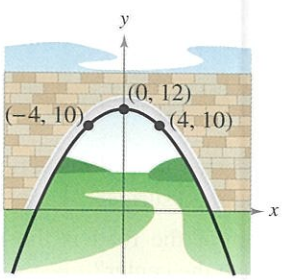


Figure for 69

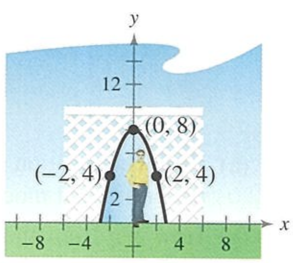
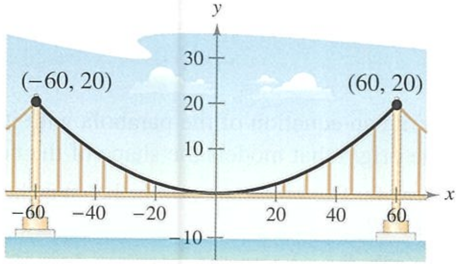


Figure for 70

70. Lattice Arch A parabolic lattice arch is 8 feet high at the vertex. At a height of 4 feet, the width of the lattice arch is 4 feet (see figure). How wide is the lattice arch at ground level?

71. Suspension Bridge Each cable of a suspension bridge is suspended (in the shape of a parabola) between two towers (see figure).



(a) Find the coordinates of the focus.

(b) Write an equation that models the cables.

72. Suspension Bridge

Each cable of the Golden Gate Bridge is suspended (in the shape of a parabola) between two towers that are 1280 meters apart. The top of each tower is 152 meters above the roadway. The cables touch the roadway at the midpoint between the towers.

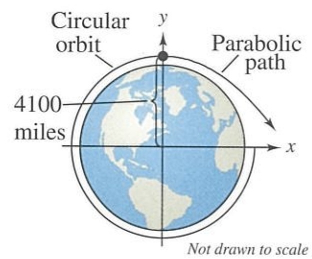
(a) Sketch the bridge on a rectangular coordinate system with the cables touching the roadway at the origin. Label the coordinates of the known points.

(b) Write an equation that models the cables.

(c) Complete the table by finding the height y of the cables over the roadway at a distance of x meters from the point where the cables touch the roadway.

Distance, x	Height, y
0	
100	
250	
400	
500	

73. Satellite Orbit A satellite in a 100-mile-high circular orbit around Earth has a velocity of approximately 17,500 miles per hour. When this velocity is multiplied by $\sqrt{2}$, the satellite has the minimum velocity necessary to escape Earth's gravity and follow a parabolic path with the center of Earth as the focus (see figure).



(a) Find the escape velocity of the satellite.

(b) Write an equation for the parabolic path of the satellite. (Assume that the radius of Earth is 4000 miles.)

74. Path of a Softball The path of a softball is modeled by

$$-12.5(y - 7.125) = (x - 6.25)^2$$

where x and y are measured in feet, with $x = 0$ corresponding to the position from which the ball was thrown.

(a) Use a graphing utility to graph the trajectory of the softball.

(b) Use the *trace* feature of the graphing utility to approximate the highest point and the range of the trajectory.

Projectile Motion **In Exercises 75 and 76, consider the path of an object projected horizontally with a velocity of v feet per second at a height of s feet, where the model for the path is**

$$x^2 = -\frac{v^2}{16}(y - s).$$

In this model (in which air resistance is disregarded), y is the height (in feet) of the projectile and x is the horizontal distance (in feet) the projectile travels.

75. A ball is thrown from the top of a 100-foot tower with a velocity of 28 feet per second.

(a) Write an equation for the parabolic path.

(b) How far does the ball travel horizontally before it strikes the ground?

76. A cargo plane is flying at an altitude of 500 feet and a speed of 255 miles per hour. A supply crate is dropped from the plane. How many *feet* will the crate travel horizontally before it hits the ground?

Exploration

True or False? **In Exercises 77–79, determine whether the statement is true or false. Justify your answer.**

77. It is possible for a parabola to intersect its directrix.

78. A tangent line to a parabola always intersects the directrix.

79. When the vertex and focus of a parabola are on a horizontal line, the directrix of the parabola is vertical.

80. Slope of a Tangent Line Let (x_1, y_1) be the coordinates of a point on the parabola $x^2 = 4py$. The equation of the line tangent to the parabola at the point is

$$y - y_1 = \frac{x_1}{2p}(x - x_1).$$

What is the slope of the tangent line?

81. Think About It Explain what each equation represents, and how equations (a) and (b) are equivalent.

(a) $y = a(x - h)^2 + k, \quad a \neq 0$

(b) $(x - h)^2 = 4p(y - k), \quad p \neq 0$

(c) $(y - k)^2 = 4p(x - h), \quad p \neq 0$

82. HOW DO YOU SEE IT?

In parts (a)–(d), describe how a plane could intersect the double-napped cone to form each conic section (see figure).

(a) Circle (b) Ellipse

(c) Parabola (d) Hyperbola

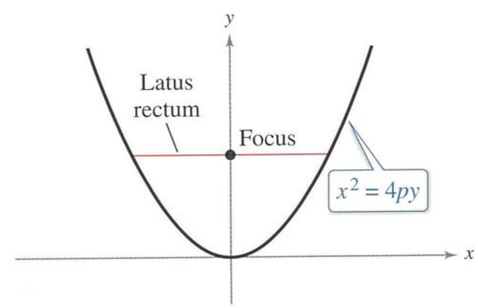
83. Think About It The graph of $x^2 + y^2 = 0$ is a degenerate conic. Sketch the graph of this equation and identify the degenerate conic. Describe the intersection of the plane and the double-napped cone for this conic.

84. Graphical Reasoning Consider the parabola $x^2 = 4py$.

(a) Use a graphing utility to graph the parabola for $p = 1$, $p = 2$, $p = 3$, and $p = 4$. Describe the effect on the graph when p increases.

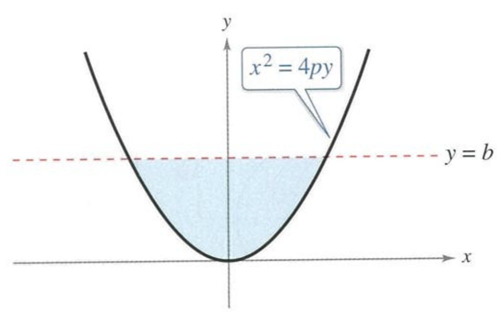
(b) Find the focus for each parabola in part (a).

(c) For each parabola in part (a), find the length of the latus rectum (see figure). How can the length of the latus rectum be determined directly from the standard form of the equation of the parabola?



(d) How can you use the result of part (c) as a sketching aid when graphing parabolas?

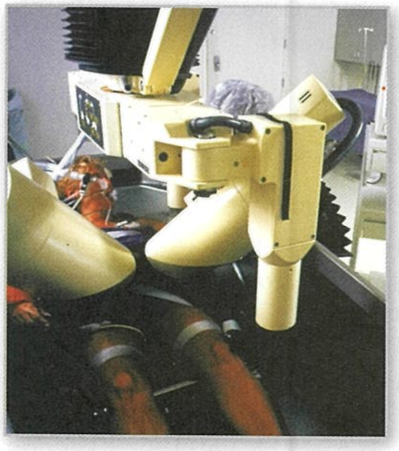
85. Geometry The area of the shaded region in the figure is $A = \frac{8}{3}p^{1/2}b^{3/2}$.



(a) Find the area when $p = 2$ and $b = 4$.

(b) Give a geometric explanation of why the area approaches 0 as p approaches 0.

10.3 Ellipses



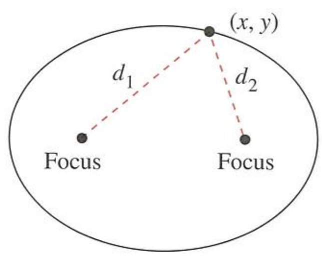
Ellipses have many real-life applications. For example, Exercise 55 on page 715 shows how a lithotripter machine uses the focal properties of an ellipse to break up kidney stones.

- **Write equations of ellipses in standard form and sketch ellipses.**
- **Use properties of ellipses to model and solve real-life problems.**
- **Find eccentricities of ellipses.**

Introduction

Another type of conic is an **ellipse.** It is defined below.

> **Definition of an Ellipse**
>
> An **ellipse** is the set of all points (x, y) in a plane, the sum of whose distances from two distinct fixed points **(foci)** is constant. See Figure 10.12.



$d_1 + d_2$ is constant.
Figure 10.12

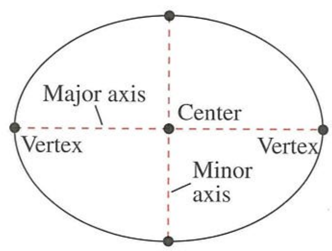
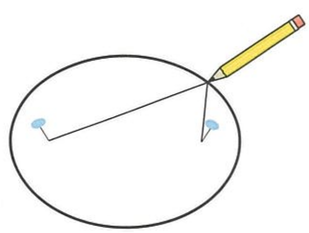
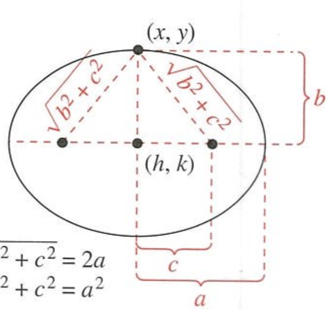


Figure 10.13

The line through the foci intersects the ellipse at two points **(vertices).** The chord joining the vertices is the **major axis,** and its midpoint is the **center** of the ellipse. The chord perpendicular to the major axis at the center is the **minor axis** of the ellipse. (See Figure 10.13.)

To visualize the definition of an ellipse, imagine two thumbtacks placed at the foci, as shown in the figure below. When the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, the path traced by the pencil is an ellipse.





$2\sqrt{b^2+c^2} = 2a$
$b^2 + c^2 = a^2$

Figure 10.14

To derive the standard form of the equation of an ellipse, consider the ellipse in Figure 10.14 with the points listed below.

Center: (h, k) Vertices: $(h \pm a, k)$ Foci: $(h \pm c, k)$

Note that the center is also the midpoint of the segment joining the foci.

The sum of the distances from any point on the ellipse to the two foci is constant. Using a vertex point, this constant sum is

$$(a + c) + (a - c) = 2a \qquad \text{Length of major axis}$$

which is the length of the major axis.

Southern Illinois University -/Getty Images

Now, if you let (x, y) be *any* point on the ellipse, then the sum of the distances between (x, y) and the two foci must also be $2a$. That is,

$$\sqrt{[x-(h-c)]^2+(y-k)^2}+\sqrt{[x-(h+c)]^2+(y-k)^2}=2a$$

which, after expanding and regrouping, reduces to

$$(a^2-c^2)(x-h)^2+a^2(y-k)^2=a^2(a^2-c^2).$$

From Figure 10.14,

$$b^2+c^2=a^2$$
$$b^2=a^2-c^2$$

which implies that the equation of the ellipse is

$$b^2(x-h)^2+a^2(y-k)^2=a^2b^2$$
$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

You would obtain a similar equation in the derivation by starting with a vertical major axis. A summary of these results is given below.

REMARK Consider the equation of the ellipse

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

If you let $a = b = r$, then the equation can be rewritten as

$$(x-h)^2+(y-k)^2=r^2$$

which is the standard form of the equation of a circle with radius r. Geometrically, when $a = b$ for an ellipse, the major and minor axes are of equal length, and so the graph is a circle.

Standard Equation of an Ellipse

The **standard form of the equation of an ellipse** with center (h, k) and major and minor axes of lengths $2a$ and $2b$, respectively, where $0 < b < a$, is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \qquad \text{Major axis is horizontal.}$$

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1. \qquad \text{Major axis is vertical.}$$

The foci lie on the major axis, c units from the center, with

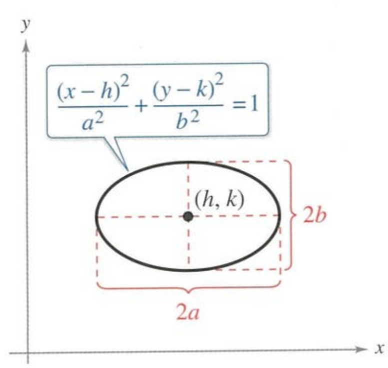
$$c^2=a^2-b^2.$$

If the center is at the origin, then the equation takes one of the forms below.

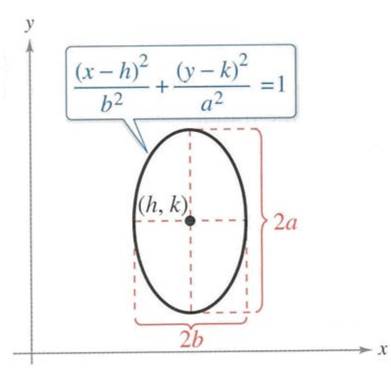
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{Major axis is horizontal.}$$

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \qquad \text{Major axis is vertical.}$$

The figures below show generalized horizontal and vertical orientations for ellipses.



Major axis is horizontal.



Major axis is vertical.

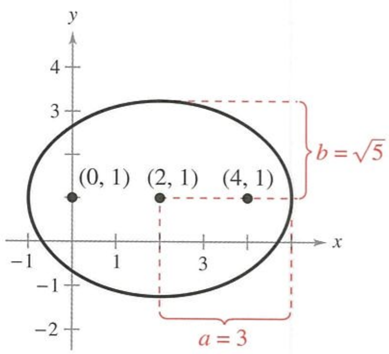


Figure 10.15

EXAMPLE 1 Finding the Standard Equation of an Ellipse

Find the standard form of the equation of the ellipse with foci (0, 1) and (4, 1) and major axis of length 6, as shown in Figure 10.15.

Solution The foci occur at (0, 1) and (4, 1), so the center of the ellipse is (2, 1) and the distance from the center to one of the foci is $c = 2$. Because $2a = 6$, you know that $a = 3$. Now, from $c^2 = a^2 - b^2$, you have

$$b = \sqrt{a^2 - c^2} = \sqrt{3^2 - 2^2} = \sqrt{5}.$$

The major axis is horizontal, so the standard form of the equation is

$$\frac{(x-2)^2}{3^2} + \frac{(y-1)^2}{(\sqrt{5})^2} = 1.$$

This equation simplifies to

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{5} = 1.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Find the standard form of the equation of the ellipse with foci (2, 0) and (2, 6) and major axis of length 8.

EXAMPLE 2 Sketching an Ellipse

Sketch the ellipse $4x^2 + y^2 = 36$ and identify the center and vertices.

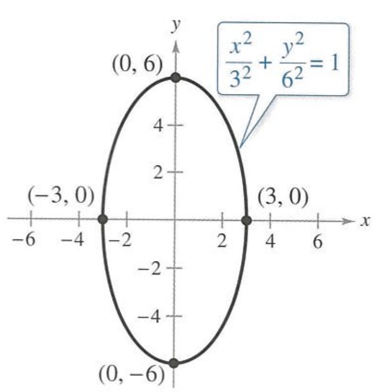
Algebraic Solution

$4x^2 + y^2 = 36$ Write original equation.

$\frac{4x^2}{36} + \frac{y^2}{36} = \frac{36}{36}$ Divide each side by 36.

$\frac{x^2}{3^2} + \frac{y^2}{6^2} = 1$ Write in standard form.

The center of the ellipse is (0, 0). The denominator of the y^2-term is greater than the denominator of the x^2-term, so the major axis is vertical. Moreover, $a^2 = 6^2$, so the endpoints of the major axis (the vertices) lie six units *up* and *down* from the center at (0, 6) and (0, −6). Similarly, the denominator of the x^2-term is $b^2 = 3^2$, so the endpoints of the minor axis (the *co-vertices*) lie three units to the *right* and *left* of the center at (3, 0) and (−3, 0). A sketch of the ellipse is at the right.



Graphical Solution

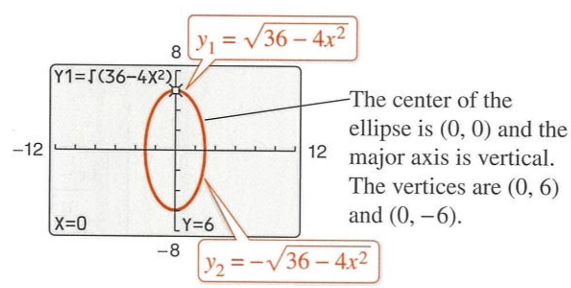
Solve the equation of the ellipse for y.

$$4x^2 + y^2 = 36$$
$$y^2 = 36 - 4x^2$$
$$y = \pm\sqrt{36 - 4x^2}$$

Then, use a graphing utility to graph

$$y_1 = \sqrt{36 - 4x^2} \quad \text{and} \quad y_2 = -\sqrt{36 - 4x^2}$$

in the same viewing window, as shown in the figure below. Be sure to use a square setting.



✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Sketch the ellipse $x^2 + 9y^2 = 81$ and identify the center and vertices.

EXAMPLE 3 Sketching an Ellipse

Find the center, vertices, and foci of the ellipse $x^2 + 4y^2 + 6x - 8y + 9 = 0$. Then sketch the ellipse.

Solution Begin by writing the original equation in standard form. In the third step, note that you add 9 and 4 to *both* sides of the equation when completing the squares.

$$x^2 + 4y^2 + 6x - 8y + 9 = 0 \quad \text{Write original equation.}$$

$$(x^2 + 6x + \square) + 4(y^2 - 2y + \square) = -9 \quad \text{Group terms and factor 4 out of } y\text{-terms.}$$

$$(x^2 + 6x + 9) + 4(y^2 - 2y + 1) = -9 + 9 + 4(1) \quad \text{Complete the squares.}$$

$$(x + 3)^2 + 4(y - 1)^2 = 4 \quad \text{Write in completed square form.}$$

$$\frac{(x + 3)^2}{2^2} + \frac{(y - 1)^2}{1^2} = 1 \quad \text{Write in standard form.}$$

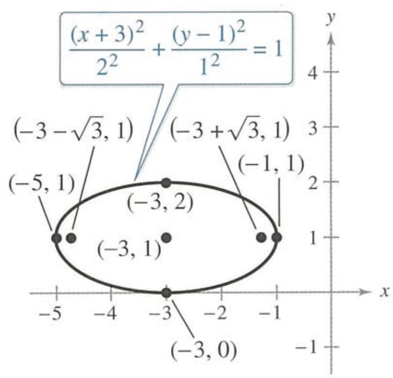


Figure 10.16

From this standard form, it follows that the center is $(h, k) = (-3, 1)$. The denominator of the x-term is $a^2 = 2^2$, so the endpoints of the major axis lie two units to the right and left of the center and the vertices are $(-1, 1)$ and $(-5, 1)$. Similarly, the denominator of the y-term is $b^2 = 1^2$, so the covertices lie one unit up and down from the center at $(-3, 2)$ and $(-3, 0)$. Now, from $c^2 = a^2 - b^2$, you have

$$c = \sqrt{2^2 - 1^2} = \sqrt{3}.$$

So, the foci of the ellipse are $(-3 + \sqrt{3}, 1)$ and $(-3 - \sqrt{3}, 1)$. Figure 10.16 shows the ellipse.

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Find the center, vertices, and foci of the ellipse $9x^2 + 4y^2 + 36x - 8y + 4 = 0$. Then sketch the ellipse.

EXAMPLE 4 Sketching an Ellipse

See LarsonPrecalculus.com for an interactive version of this type of example.

Find the center, vertices, and foci of the ellipse $4x^2 + y^2 - 8x + 4y - 8 = 0$. Then sketch the ellipse.

Solution Complete the square to write the original equation in standard form.

$$4x^2 + y^2 - 8x + 4y - 8 = 0 \quad \text{Write original equation.}$$

$$4(x^2 - 2x + \square) + (y^2 + 4y + \square) = 8 \quad \text{Group terms and factor 4 out of } x\text{-terms.}$$

$$4(x^2 - 2x + 1) + (y^2 + 4y + 4) = 8 + 4(1) + 4 \quad \text{Complete the squares.}$$

$$4(x - 1)^2 + (y + 2)^2 = 16 \quad \text{Write in completed square form.}$$

$$\frac{(x - 1)^2}{2^2} + \frac{(y + 2)^2}{4^2} = 1 \quad \text{Write in standard form.}$$

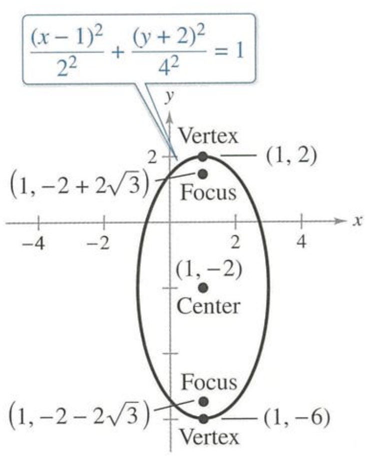


Figure 10.17

The major axis is vertical, $h = 1$, $k = -2$, $a = 4$, $b = 2$, and

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 4} = \sqrt{12} = 2\sqrt{3}.$$

So, the center is $(h, k) = (1, -2)$, the vertices are $(1, -6)$ and $(1, 2)$, and the foci are $(1, -2 - 2\sqrt{3})$ and $(1, -2 + 2\sqrt{3})$. Figure 10.17 shows the ellipse.

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Find the center, vertices, and foci of the ellipse $5x^2 + 9y^2 + 10x - 54y + 41 = 0$. Then sketch the ellipse.

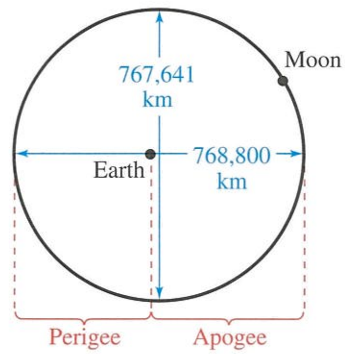
Application

Ellipses have many practical and aesthetic uses. For example, machine gears, supporting arches, and acoustic designs often involve elliptical shapes. The orbits of satellites and planets are also ellipses. Example 5 investigates the elliptical orbit of the moon about Earth.

EXAMPLE 5 An Application Involving an Elliptical Orbit

The moon travels about Earth in an elliptical orbit with the center of Earth at one focus, as shown in the figure at the right. The major and minor axes of the orbit have lengths of 768,800 kilometers and 767,641 kilometers, respectively. Find the greatest and least distances (the *apogee* and *perigee,* respectively) from Earth's center to the moon's center. Then use a graphing utility to graph the orbit of the moon.

REMARK Note in Example 5 that Earth is *not* the center of the moon's orbit.



In Exercise 56, you will investigate the elliptical orbit of Halley's comet about the sun. Halley's comet is visible from Earth approximately every 76.1 years. The comet's latest appearance was in 1986.

Solution Because $2a = 768{,}800$ and $2b = 767{,}641$, you have $a = 384{,}400$ and $b = 383{,}820.5$, which implies that

$$c = \sqrt{a^2 - b^2} = \sqrt{384{,}400^2 - 383{,}820.5^2} \approx 21{,}099.$$

So, the greatest distance between the center of Earth and the center of the moon is

$$a + c \approx 384{,}400 + 21{,}099 = 405{,}499 \text{ kilometers}$$

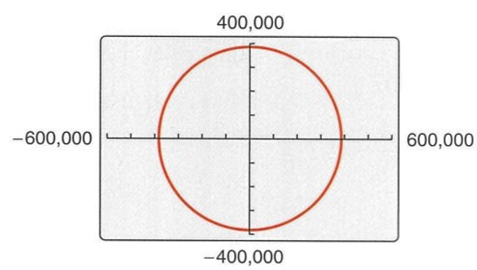
and the least distance is

$$a - c \approx 384{,}400 - 21{,}099 = 363{,}301 \text{ kilometers.}$$

To use a graphing utility to graph the orbit of the moon, first let $a = 384{,}400$ and $b = 383{,}820.5$ in the standard form of an equation of an ellipse centered at the origin, and then solve for y.

$$\frac{x^2}{384{,}400^2} + \frac{y^2}{383{,}820.5^2} = 1 \quad \Longrightarrow \quad y = \pm 383{,}820.5\sqrt{1 - \frac{x^2}{384{,}400^2}}$$

Graph the upper and lower portions in the same viewing window, as shown below.



✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Encke's comet travels about the sun in an elliptical orbit with the center of the sun at one focus. The major and minor axes of the orbit have lengths of approximately 4.420 astronomical units and 2.356 astronomical units, respectively. (An astronomical unit is about 93 million miles.) Find the greatest and least distances (the *aphelion* and *perihelion,* respectively). from the sun's center to the comet's center. Then use a graphing utility to graph the orbit of the comet.

Eccentricity

It was difficult for early astronomers to detect that the orbits of the planets are ellipses because the foci of the planetary orbits are relatively close to their centers, and so the orbits are nearly circular. You can measure the "ovalness" of an ellipse by using the concept of **eccentricity.**

Definition of Eccentricity

The **eccentricity** e of an ellipse is the ratio $e = \dfrac{c}{a}$.

Note that $0 < e < 1$ for *every* ellipse.

To see how this ratio describes the shape of an ellipse, note that because the foci of an ellipse are located along the major axis between the vertices and the center, it follows that $0 < c < a$. For an ellipse that is nearly circular, the foci are close to the center and the ratio c/a is close to 0, as shown in Figure 10.18. On the other hand, for an elongated ellipse, the foci are close to the vertices and the ratio c/a is close to 1, as shown in Figure 10.19.

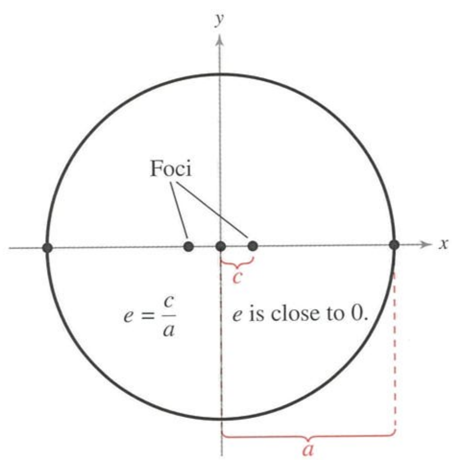


Figure 10.18

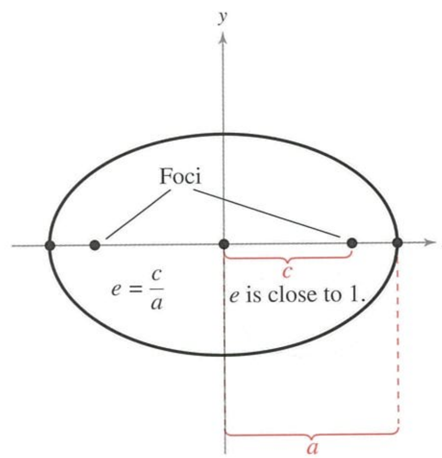


Figure 10.19

The orbit of the moon has an eccentricity of $e \approx 0.0549$. The eccentricities of the eight planetary orbits are listed below.

Mercury:	$e \approx 0.2056$	Jupiter:	$e \approx 0.0489$
Venus:	$e \approx 0.0067$	Saturn:	$e \approx 0.0565$
Earth:	$e \approx 0.0167$	Uranus:	$e \approx 0.0457$
Mars:	$e \approx 0.0935$	Neptune:	$e \approx 0.0113$

The time it takes Saturn to orbit the sun is about 29.5 Earth years.

Summarize (Section 10.3)

1. State the definition of an ellipse and the standard form of the equation of an ellipse *(page 708)*. For examples involving the equations and graphs of ellipses, see Examples 1–4.
2. Describe a real-life application of an ellipse *(page 712, Example 5)*.
3. State the definition of the eccentricity of an ellipse and explain how eccentricity describes the shape of an ellipse *(page 713)*.

10.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

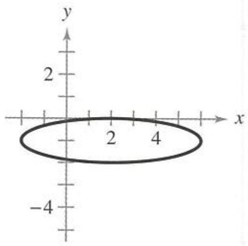
Vocabulary: Fill in the blanks.

1. An ________ is the set of all points (x, y) in a plane, the sum of whose distances from two distinct fixed points, called ________, is constant.
2. The chord joining the vertices of an ellipse is the ________ ________, and its midpoint is the ________ of the ellipse.
3. The chord perpendicular to the major axis at the center of an ellipse is the ________ ________ of the ellipse.
4. You can measure the "ovalness" of an ellipse by using the concept of ________.

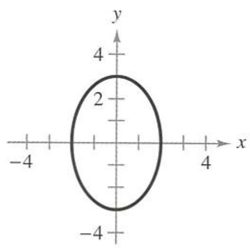
Skills and Applications

Matching **In Exercises 5–8, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

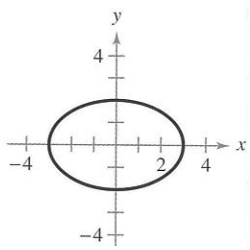
(a)



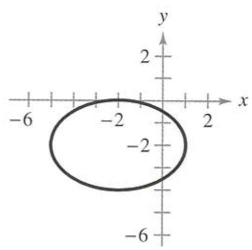
(b)



(c)



(d)



5. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

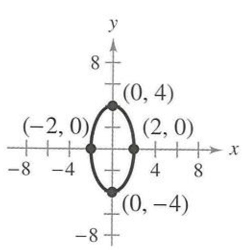
6. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

7. $\dfrac{(x-2)^2}{16} + (y+1)^2 = 1$

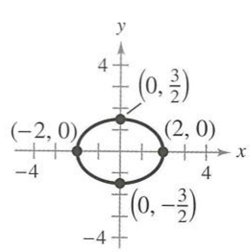
8. $\dfrac{(x+2)^2}{9} + \dfrac{(y+2)^2}{4} = 1$

An Ellipse Centered at the Origin **In Exercises 9–18, find the standard form of the equation of the ellipse with the given characteristics and center at the origin.**

9.



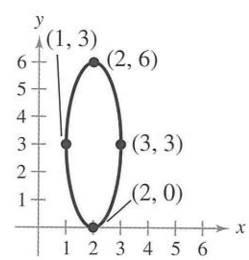
10.



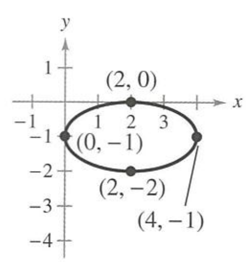
11. Vertices: $(\pm 7, 0)$; foci: $(\pm 2, 0)$
12. Vertices: $(0, \pm 8)$; foci: $(0, \pm 4)$
13. Foci: $(\pm 4, 0)$; major axis of length 10
14. Foci: $(0, \pm 3)$; major axis of length 8
15. Vertical major axis; passes through the points $(0, 6)$ and $(3, 0)$
16. Horizontal major axis; passes through the points $(5, 0)$ and $(0, 2)$
17. Vertices: $(\pm 6, 0)$; passes through the point $(4, 1)$
18. Vertices: $(0, \pm 8)$; passes through the point $(3, 4)$

Finding the Standard Equation of an Ellipse **In Exercises 19–30, find the standard form of the equation of the ellipse with the given characteristics.**

19.



20.



21. Vertices: $(2, 0)$, $(10, 0)$; minor axis of length 4
22. Vertices: $(3, 1)$, $(3, 11)$; minor axis of length 2
23. Foci: $(0, 0)$, $(4, 0)$; major axis of length 6
24. Foci: $(0, 0)$, $(0, 8)$; major axis of length 16
25. Center: $(1, 3)$; vertex: $(-2, 3)$; minor axis of length 4
26. Center: $(2, -1)$; vertex: $\left(2, \frac{1}{2}\right)$; minor axis of length 2
27. Center: $(1, 4)$; $a = 2c$; vertices: $(1, 0)$, $(1, 8)$
28. Center: $(3, 2)$; $a = 3c$; foci: $(1, 2)$, $(5, 2)$
29. Vertices: $(0, 2)$, $(4, 2)$; endpoints of the minor axis: $(2, 3)$, $(2, 1)$
30. Vertices: $(5, 0)$, $(5, 12)$; endpoints of the minor axis: $(1, 6)$, $(9, 6)$

Sketching an Ellipse **In Exercises 31–46, find the center, vertices, foci, and eccentricity of the ellipse. Then sketch the ellipse.**

31. $\frac{x^2}{25} + \frac{y^2}{16} = 1$

32. $\frac{x^2}{16} + \frac{y^2}{81} = 1$

33. $9x^2 + y^2 = 36$

34. $x^2 + 16y^2 = 64$

35. $\frac{(x-4)^2}{16} + \frac{(y+1)^2}{25} = 1$

36. $\frac{(x+3)^2}{12} + \frac{(y-2)^2}{16} = 1$

37. $\frac{(x+5)^2}{9/4} + (y-1)^2 = 1$

38. $(x+2)^2 + \frac{(y+4)^2}{1/4} = 1$

39. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

40. $9x^2 + 4y^2 - 54x + 40y + 37 = 0$

41. $x^2 + 5y^2 - 8x - 30y - 39 = 0$

42. $3x^2 + y^2 + 18x - 2y - 8 = 0$

43. $6x^2 + 2y^2 + 18x - 10y + 2 = 0$

44. $x^2 + 4y^2 - 6x + 20y - 2 = 0$

45. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

46. $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

Graphing an Ellipse **In Exercises 47–50, use a graphing utility to graph the ellipse. Find the center, foci, and vertices.**

47. $5x^2 + 3y^2 = 15$

48. $3x^2 + 4y^2 = 12$

49. $x^2 + 9y^2 - 10x + 36y + 52 = 0$

50. $4x^2 + 3y^2 - 8x + 18y + 19 = 0$

51. **Using Eccentricity** Find an equation of the ellipse with vertices $(\pm 5, 0)$ and eccentricity $e = \frac{4}{5}$.

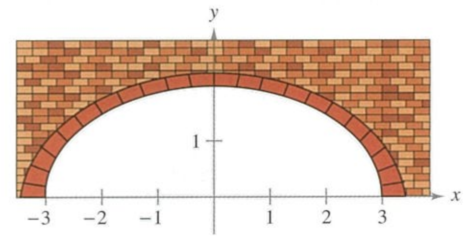
52. **Using Eccentricity** Find an equation of the ellipse with vertices $(0, \pm 8)$ and eccentricity $e = \frac{1}{2}$.

53. **Architecture** Statuary Hall is an elliptical room in the United States Capitol in Washington, D.C. The room is also called the Whispering Gallery because a person standing at one focus of the room can hear even a whisper spoken by a person standing at the other focus. The dimensions of Statuary Hall are 46 feet wide by 97 feet long.

(a) Find an equation of the shape of the room.

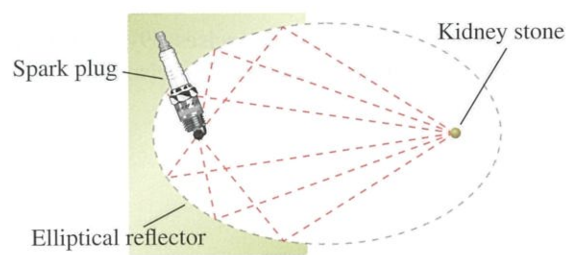
(b) Determine the distance between the foci.

54. **Architecture** A mason is building a semielliptical fireplace arch that has a height of 2 feet at the center and a width of 6 feet along the base (see figure). The mason draws the semiellipse on the wall by the method shown on page 708. Find the positions of the thumbtacks and the length of the string.



55. **Lithotripter**

A lithotripter machine uses an elliptical reflector to break up kidney stones nonsurgically. A spark plug in the reflector generates energy waves at one focus of an ellipse. The reflector directs these waves toward the kidney stone, positioned at the other focus of the ellipse, with enough energy to break up the stone (see figure). The lengths of the major and minor axes of the ellipse are 280 millimeters and 160 millimeters, respectively. How far is the spark plug from the kidney stone?



56. **Astronomy** Halley's comet has an elliptical orbit with the center of the sun at one focus. The eccentricity of the orbit is approximately 0.967. The length of the major axis of the orbit is approximately 35.88 astronomical units. (An astronomical unit is about 93 million miles.)

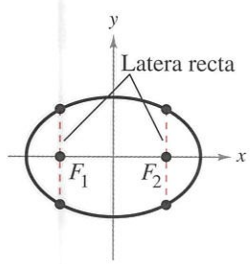
(a) Find an equation of the orbit. Place the center of the orbit at the origin and place the major axis on the x-axis.

(b) Use a graphing utility to graph the equation of the orbit.

(c) Find the greatest and least distances (the aphelion and perihelion, respectively) from the sun's center to the comet's center.

57. Astronomy The first artificial satellite to orbit Earth was Sputnik I (launched by the former Soviet Union in 1957). Its highest point above Earth's surface was 939 kilometers, and its lowest point was 215 kilometers (see figure). The center of Earth was at one focus of the elliptical orbit. Find the eccentricity of the orbit. (Assume the radius of Earth is 6378 kilometers.)

58. Geometry A line segment through a focus of an ellipse with endpoints on the ellipse and perpendicular to the major axis is called a **latus rectum** of the ellipse. An ellipse has two latera recta. Knowing the length of the latera recta is helpful in sketching an ellipse because it yields other points on the curve (see figure). Show that the length of each latus rectum is $2b^2/a$.



Using Latera Recta **In Exercises 59–62, sketch the ellipse using the latera recta (see Exercise 58).**

59. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

60. $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$

61. $5x^2 + 3y^2 = 15$

62. $9x^2 + 4y^2 = 36$

Exploration

True or False? **In Exercises 63 and 64, determine whether the statement is true or false. Justify your answer.**

63. The graph of $x^2 + 4y^4 - 4 = 0$ is an ellipse.

64. It is easier to distinguish the graph of an ellipse from the graph of a circle when the eccentricity of the ellipse is close to 1.

65. Think About It Find an equation of an ellipse such that for any point on the ellipse, the sum of the distances from the point to the points $(2, 2)$ and $(10, 2)$ is 36.

66. Think About It At the beginning of this section, you learned that an ellipse can be drawn using two thumbtacks, a string of fixed length (greater than the distance between the two thumbtacks), and a pencil. When the ends of the string are fastened to the thumbtacks and the string is drawn taut with the pencil, the path traced by the pencil is an ellipse.

(a) What is the length of the string in terms of a?

(b) Explain why the path is an ellipse.

67. Conjecture Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a + b = 20.$$

(a) The area of the ellipse is given by $A = \pi ab$. Write the area of the ellipse as a function of a.

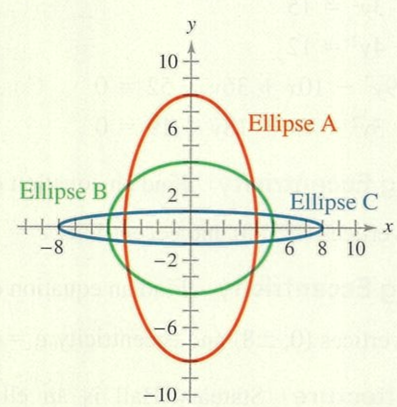
(b) Find the equation of an ellipse with an area of 264 square centimeters.

(c) Complete the table using your equation from part (a). Then make a conjecture about the shape of the ellipse with maximum area.

a	8	9	10	11	12	13
A						

(d) Use a graphing utility to graph the area function and use the graph to support your conjecture in part (c).

68. HOW DO YOU SEE IT? Without performing any calculations, order the eccentricities of the ellipses from least to greatest.



69. Proof Show that $a^2 = b^2 + c^2$ for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a > 0$, $b > 0$, and the distance from the center of the ellipse $(0, 0)$ to a focus is c.

10.4 Hyperbolas

Hyperbolas have many types of real-life applications. For example, in Exercise 53 on page 725, you will investigate the use of hyperbolas in long distance radio navigation for aircraft and ships.

- Write equations of hyperbolas in standard form.
- Find asymptotes of and sketch hyperbolas.
- Use properties of hyperbolas to solve real-life problems.
- Classify conics from their general equations.

Introduction

The definition of a **hyperbola** is similar to that of an ellipse. For an ellipse, the *sum* of the distances between the foci and a point on the ellipse is constant. For a hyperbola, the absolute value of the *difference* of the distances between the foci and a point on the hyperbola is constant.

> **Definition of a Hyperbola**
>
> A **hyperbola** is the set of all points (x, y) in a plane for which the absolute value of the difference of the distances from two distinct fixed points **(foci)** is constant. See Figure 10.20.

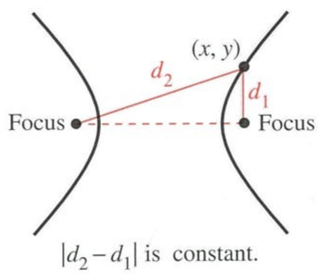


Figure 10.20

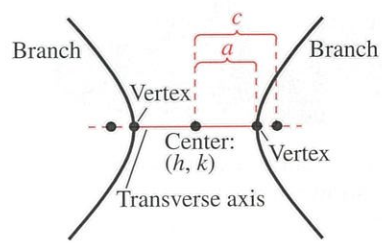


Figure 10.21

The graph of a hyperbola has two disconnected parts **(branches).** The line through the foci intersects the hyperbola at two points **(vertices).** The line segment connecting the vertices is the **transverse axis,** and its midpoint is the **center** of the hyperbola.

Consider the hyperbola in Figure 10.21 with the points listed below.

Center: (h, k) Vertices: $(h \pm a, k)$ Foci: $(h \pm c, k)$

Note that the center is also the midpoint of the segment joining the foci.

The absolute value of the difference of the distances from *any* point on the hyperbola to the two foci is constant. Using a vertex point, this constant value is

$$|[2a + (c - a)] - (c - a)| = |2a| = 2a \qquad \text{Length of transverse axis}$$

which is the length of the transverse axis. Now, if you let (x, y) be *any* point on the hyperbola, then

$$|d_2 - d_1| = 2a$$

(see Figure 10.20). You would obtain the same result for a hyperbola with a vertical transverse axis.

The development of the standard form of the equation of a hyperbola is similar to that of an ellipse. Note in the definition on the next page that a, b, and c are related differently for hyperbolas than for ellipses. For a hyperbola, the distance between the foci and the center is greater than the distance between the vertices and the center.

Standard Equation of a Hyperbola

The **standard form of the equation of a hyperbola** with center (h, k) is

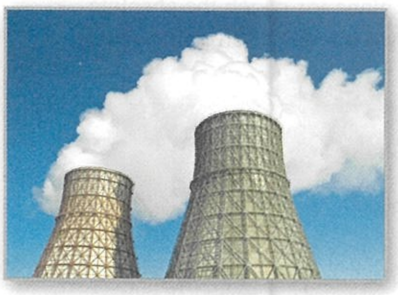
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \qquad \text{Transverse axis is vertical.}$$

The vertices are a units from the center, and the foci are c units from the center. Moreover, $c^2 = a^2 + b^2$. If the center is at the origin, then the equation takes one of the forms below.

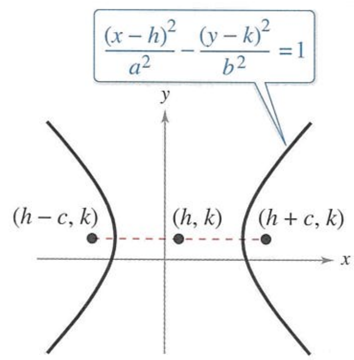
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \text{Transverse axis is vertical.}$$

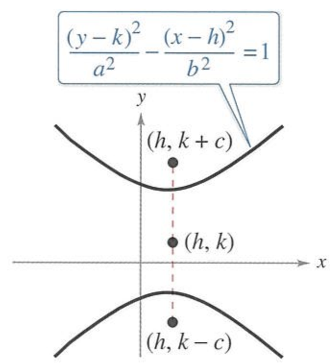


Nuclear cooling towers such as those shown above are in the shapes of hyperboloids. The vertical cross sections of these cooling towers are hyperbolas.

The figures below show generalized horizontal and vertical orientations for hyperbolas.



Transverse axis is horizontal.



Transverse axis is vertical.

EXAMPLE 1 Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola with vertices $(0, 2)$ and $(4, 2)$ and foci $(-1, 2)$ and $(5, 2)$, as shown in Figure 10.22.

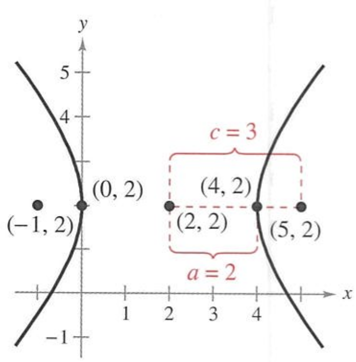


Figure 10.22

Solution The foci occur at $(-1, 2)$ and $(5, 2)$, so the center of the hyperbola is $(2, 2)$. Furthermore, $c = 5 - 2 = 3$ and $a = 4 - 2 = 2$, and it follows that

$$b = \sqrt{c^2 - a^2} = \sqrt{3^2 - 2^2} = \sqrt{9 - 4} = \sqrt{5}.$$

The hyperbola has a horizontal transverse axis, so the standard form of the equation is

$$\frac{(x-2)^2}{2^2} - \frac{(y-2)^2}{(\sqrt{5})^2} = 1.$$

This equation simplifies to

$$\frac{(x-2)^2}{4} - \frac{(y-2)^2}{5} = 1.$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Find the standard form of the equation of the hyperbola with vertices $(2, -4)$ and $(2, 2)$ and foci $(2, -5)$ and $(2, 3)$.

Asymptotes of a Hyperbola

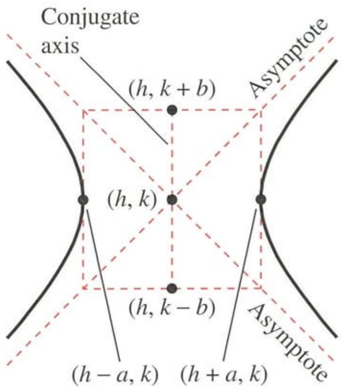


Figure 10.23

Every hyperbola has two *asymptotes* that intersect at the center of the hyperbola, as shown in Figure 10.23. The asymptotes pass through the vertices of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k). The **conjugate axis** of a hyperbola is the line segment of length $2b$ joining $(h, k + b)$ and $(h, k - b)$ when the transverse axis is horizontal (as in Figure 10.23), and joining $(h + b, k)$ and $(h - b, k)$ when the transverse axis is vertical.

Asymptotes of a Hyperbola

The equations of the asymptotes of a hyperbola are

$$y = k \pm \frac{b}{a}(x - h) \qquad \text{Asymptotes for horizontal transverse axis}$$

$$y = k \pm \frac{a}{b}(x - h). \qquad \text{Asymptotes for vertical transverse axis}$$

EXAMPLE 2 Sketching a Hyperbola

Sketch the hyperbola $4x^2 - y^2 = 16$.

Algebraic Solution

Divide each side of the original equation by 16, and write the equation in standard form.

$$\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1 \qquad \text{Write in standard form.}$$

The center of the hyperbola is $(0, 0)$. The x^2-term is postive, so the transverse axis is horizontal. The vertices occur at $(-2, 0)$ and $(2, 0)$, and the endpoints of the conjugate axis occur at $(0, -4)$ and $(0, 4)$. Use the vertices and the endpoints of the conjugate axis to sketch the rectangle shown in Figure 10.24. Sketch the asymptotes $y = 2x$ and $y = -2x$ through the opposite corners of the rectangle. Now, from $c^2 = a^2 + b^2$, you have $c = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}$. So, the foci of the hyperbola are $(-2\sqrt{5}, 0)$ and $(2\sqrt{5}, 0)$. Figure 10.25 shows a sketch of the hyperbola.

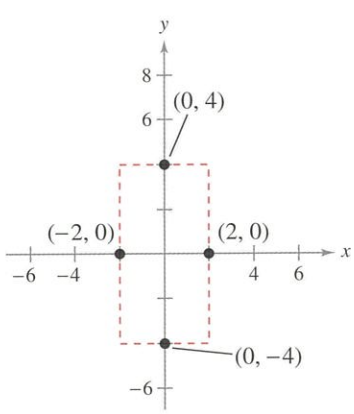


Figure 10.24

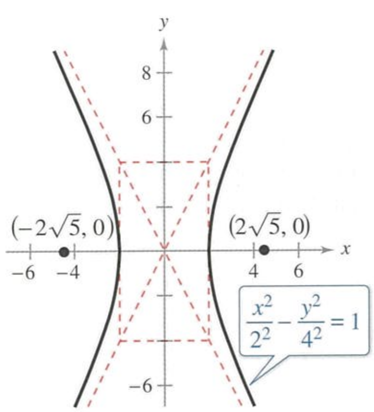


Figure 10.25

Graphical Solution

Solve the equation of the hyperbola for y.

$$4x^2 - y^2 = 16$$

$$4x^2 - 16 = y^2$$

$$\pm\sqrt{4x^2 - 16} = y$$

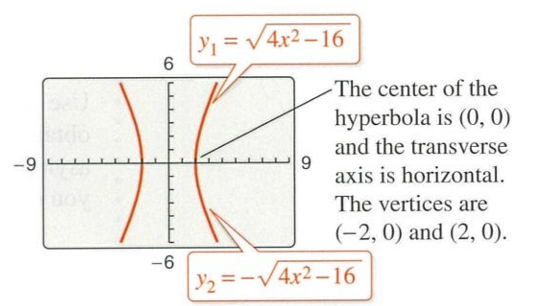
Then use a graphing utility to graph

$$y_1 = \sqrt{4x^2 - 16}$$

and

$$y_2 = -\sqrt{4x^2 - 16}$$

in the same viewing window, as shown in the figure below. Be sure to use a square setting.



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Sketch the hyperbola $4y^2 - 9x^2 = 36$.

EXAMPLE 3 Sketching a Hyperbola

Sketch the hyperbola $4x^2 - 3y^2 + 8x + 16 = 0$.

Solution

$$4x^2 - 3y^2 + 8x + 16 = 0 \qquad \text{Write original equation.}$$

$$(4x^2 + 8x) - 3y^2 = -16 \qquad \text{Group terms.}$$

$$4(x^2 + 2x) - 3y^2 = -16 \qquad \text{Factor 4 out of } x\text{-terms.}$$

$$4(x^2 + 2x + 1) - 3y^2 = -16 + 4(1) \qquad \text{Complete the square.}$$

$$4(x + 1)^2 - 3y^2 = -12 \qquad \text{Write in completed square form.}$$

$$-\frac{(x + 1)^2}{3} + \frac{y^2}{4} = 1 \qquad \text{Divide each side by } -12.$$

$$\frac{y^2}{2^2} - \frac{(x + 1)^2}{(\sqrt{3})^2} = 1 \qquad \text{Write in standard form.}$$

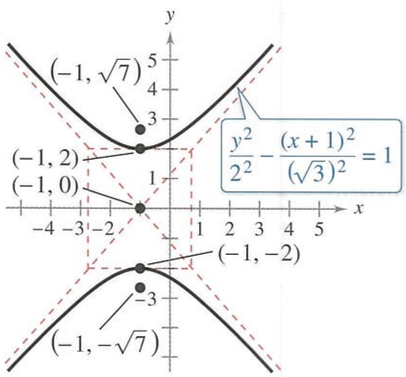


Figure 10.26

The center of the hyperbola is $(-1, 0)$. The y^2-term is positive, so the transverse axis is vertical. The vertices occur at $(-1, 2)$ and $(-1, -2)$, and the endpoints of the conjugate axis occur at $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$. Draw a rectangle through the vertices and the endpoints of the conjugate axes. Sketch the asymptotes by drawing lines through the opposite corners of the rectangle. Using $a = 2$ and $b = \sqrt{3}$, the equations of the asymptotes are

$$y = \frac{2}{\sqrt{3}}(x + 1) \quad \text{and} \quad y = -\frac{2}{\sqrt{3}}(x + 1).$$

Finally, from $c^2 = a^2 + b^2$, you have $c = \sqrt{2^2 + (\sqrt{3})^2} = \sqrt{7}$. So, the foci of the hyperbola are $(-1, \sqrt{7})$ and $(-1, -\sqrt{7})$. Figure 10.26 shows a sketch of the hyperbola.

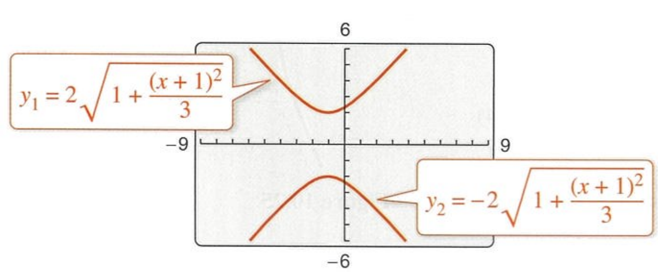
Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Sketch the hyperbola $9x^2 - 4y^2 + 8y - 40 = 0$.

TECHNOLOGY To use a graphing utility to graph a hyperbola, graph the upper and lower portions in the same viewing window. For instance, to graph the hyperbola in Example 3, first solve for y to get

$$y_1 = 2\sqrt{1 + \frac{(x + 1)^2}{3}} \quad \text{and} \quad y_2 = -2\sqrt{1 + \frac{(x + 1)^2}{3}}.$$

Use a viewing window in which $-9 \le x \le 9$ and $-6 \le y \le 6$. You should obtain the graph shown below. Notice that the graphing utility does not draw the asymptotes. However, by graphing the asymptotes in the same viewing window, you can see that the values of the hyperbola approach the asymptotes.



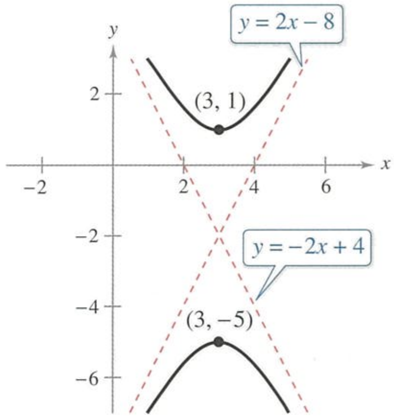


Figure 10.27

EXAMPLE 4 Using Asymptotes to Find the Standard Equation

See LarsonPrecalculus.com for an interactive version of this type of example.

Find the standard form of the equation of the hyperbola with vertices $(3, -5)$ and $(3, 1)$ and asymptotes

$$y = 2x - 8 \quad \text{and} \quad y = -2x + 4$$

as shown in Figure 10.27.

Solution The center of the hyperbola is $(3, -2)$. Furthermore, the hyperbola has a vertical transverse axis with $a = 3$. The slopes of the asymptotes are

$$m_1 = 2 = \frac{a}{b} \quad \text{and} \quad m_2 = -2 = -\frac{a}{b}$$

and $a = 3$, so

$$2 = \frac{a}{b} \Rightarrow 2 = \frac{3}{b} \Rightarrow b = \frac{3}{2}.$$

The standard form of the equation of the hyperbola is

$$\frac{(y+2)^2}{3^2} - \frac{(x-3)^2}{\left(\frac{3}{2}\right)^2} = 1.$$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Find the standard form of the equation of the hyperbola with vertices $(3, 2)$ and $(9, 2)$ and asymptotes

$$y = -2 + \frac{2}{3}x \quad \text{and} \quad y = 6 - \frac{2}{3}x.$$

As with ellipses, the *eccentricity* of a hyperbola is

$$e = \frac{c}{a}. \qquad \text{Eccentricity}$$

You know that $c > a$ for a hyperbola, so it follows that $e > 1$. When the eccentricity is large, the branches of the hyperbola are nearly flat, as shown in Figure 10.28. When the eccentricity is close to 1, the branches of the hyperbola are more narrow, as shown in Figure 10.29.

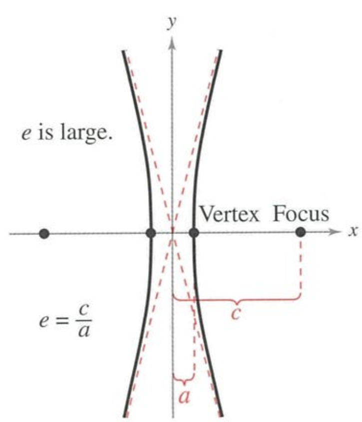


Figure 10.28

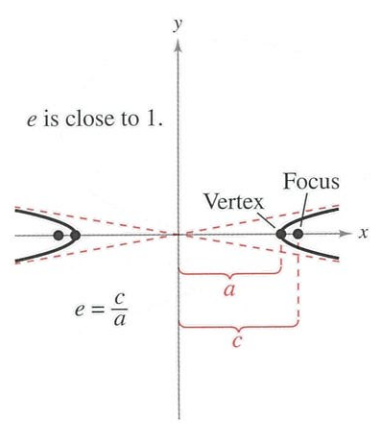


Figure 10.29

Applications

The next example shows how the properties of hyperbolas are used in radar and other detection systems. The United States and Great Britain developed this application during World War II.

EXAMPLE 5 An Application Involving Hyperbolas

Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds before microphone B. Where did the explosion occur?

Solution Assuming sound travels at 1100 feet per second, you know that the explosion took place 2200 feet farther from B than from A, as shown in the figure. The locus of all points that are 2200 feet closer to A than to B is one branch of the hyperbola of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

$$a = \frac{2200}{2} = 1100.$$

Because

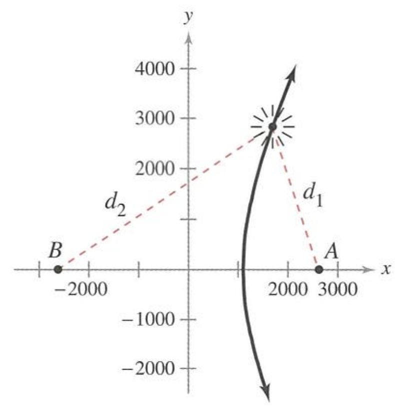
$$c = \frac{5280}{2} = 2640$$

it follows that

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= 2640^2 - 1100^2 \\ &= 5{,}759{,}600. \end{aligned}$$

So, the explosion occurred somewhere on the right branch of the hyperbola

$$\frac{x^2}{1{,}210{,}000} - \frac{y^2}{5{,}759{,}600} = 1.$$



$2c = 1 \text{ mi} = 5280 \text{ ft}$
$|d_2 - d_1| = 2a = 2200 \text{ ft}$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Repeat Example 5 when microphone A receives the sound 4 seconds before microphone B.

Another interesting application of conic sections involves the orbits of comets in our solar system. Comets can have elliptical, parabolic, or hyperbolic orbits. The center of the sun is a focus of each of these orbits, and each orbit has a vertex at the point where the comet is closest to the sun, as shown in Figure 10.30. Undoubtedly, many comets with parabolic or hyperbolic orbits have not been identified. You get to see such comets only *once*. Comets with elliptical orbits, such as Halley's comet, are the only ones that remain in our solar system.

If p is the distance between the vertex and the focus (in meters), and v is the speed of the comet at the vertex (in meters per second), then the type of orbit is determined as follows, where $M = 1.989 \times 10^{30}$ kilograms (the mass of the sun) and $G \approx 6.67 \times 10^{-11}$ cubic meter per kilogram-second squared (the universal gravitational constant).

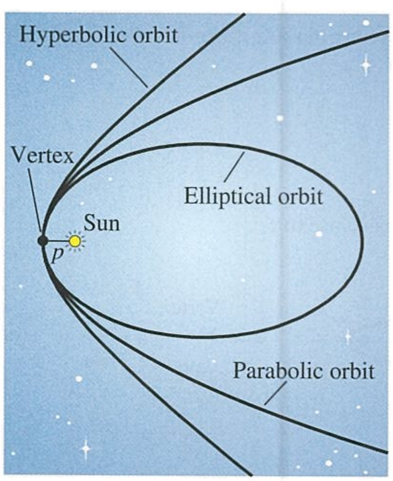


Figure 10.30

1. Elliptical: $v < \sqrt{2GM/p}$
2. Parabolic: $v = \sqrt{2GM/p}$
3. Hyperbolic: $v > \sqrt{2GM/p}$

General Equations of Conics

Classifying a Conic from Its General Equation

The graph of $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is one of the following.

1. *Circle:* $A = C$ — $A \neq 0$
2. *Parabola:* $AC = 0$ — $A = 0$ or $C = 0$, but not both.
3. *Ellipse:* $AC > 0$ — $A \neq C$ and A and C have like signs.
4. *Hyperbola:* $AC < 0$ — A and C have unlike signs.

The test above is valid when the graph is a conic. The test does not apply to equations such as $x^2 + y^2 = -1$, whose graph is not a conic.

EXAMPLE 6 Classifying Conics from General Equations

a. For the equation $4x^2 - 9x + y - 5 = 0$, you have

$AC = 4(0) = 0.$ Parabola

So, the graph is a parabola.

b. For the equation $4x^2 - y^2 + 8x - 6y + 4 = 0$, you have

$AC = 4(-1) < 0.$ Hyperbola

So, the graph is a hyperbola.

c. For the equation $2x^2 + 4y^2 - 4x + 12y = 0$, you have

$AC = 2(4) > 0.$ Ellipse

So, the graph is an ellipse.

d. For the equation $2x^2 + 2y^2 - 8x + 12y + 2 = 0$, you have

$A = C = 2.$ Circle

So, the graph is a circle.

Caroline Herschel (1750–1848) was the first woman to be credited with discovering a comet. During her long life, this German astronomer discovered a total of eight comets.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Classify the graph of each equation.

a. $3x^2 + 3y^2 - 6x + 6y + 5 = 0$

b. $2x^2 - 4y^2 + 4x + 8y - 3 = 0$

c. $3x^2 + y^2 + 6x - 2y + 3 = 0$

d. $2x^2 + 4x + y - 2 = 0$

Summarize (Section 10.4)

1. State the definition of a hyperbola and the standard form of the equation of a hyperbola *(page 717)*. For an example of finding the standard form of the equation of a hyperbola, see Example 1.
2. Explain how to find asymptotes of and sketch a hyperbola *(page 719)*. For examples involving asymptotes and graphs of hyperbolas, see Examples 2–4.
3. Describe a real-life application of a hyperbola *(page 722, Example 5)*.
4. Explain how to classify a conic from its general equation *(page 723)*. For an example of classifying conics from their general equations, see Example 6.

10.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

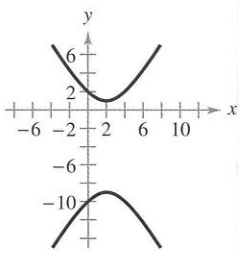
Vocabulary: Fill in the blanks.

1. A ________ is the set of all points (x, y) in a plane for which the absolute value of the difference of the distances from two distinct fixed points, called ________, is constant.
2. The graph of a hyperbola has two disconnected parts called ________.
3. The line segment connecting the vertices of a hyperbola is the ________ ________, and its midpoint is the ________ of the hyperbola.
4. Every hyperbola has two ________ that intersect at the center of the hyperbola.

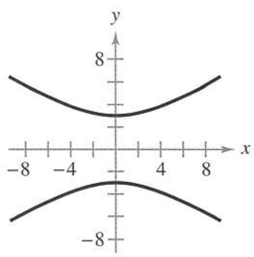
Skills and Applications

Matching **In Exercises 5–8, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

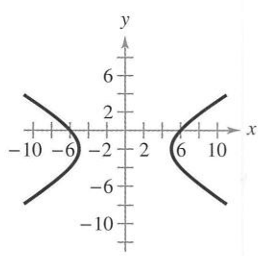
(a)



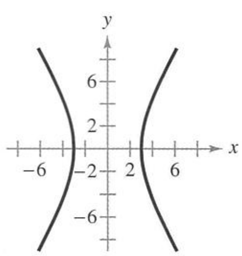
(b)



(c)



(d)



5. $\dfrac{y^2}{9} - \dfrac{x^2}{25} = 1$
6. $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$
7. $\dfrac{x^2}{25} - \dfrac{(y+2)^2}{9} = 1$
8. $\dfrac{(y+4)^2}{25} - \dfrac{(x-2)^2}{9} = 1$

Finding the Standard Equation of a Hyperbola **In Exercises 9–18, find the standard form of the equation of the hyperbola with the given characteristics.**

9. Vertices: $(0, \pm 2)$; foci: $(0, \pm 4)$
10. Vertices: $(\pm 4, 0)$; foci: $(\pm 6, 0)$
11. Vertices: $(2, 0)$, $(6, 0)$; foci: $(0, 0)$, $(8, 0)$
12. Vertices: $(2, 3)$, $(2, -3)$; foci: $(2, 6)$, $(2, -6)$
13. Vertices: $(4, 1)$, $(4, 9)$; foci: $(4, 0)$, $(4, 10)$
14. Vertices: $(-1, 1)$, $(3, 1)$; foci: $(-2, 1)$, $(4, 1)$
15. Vertices: $(2, 3)$, $(2, -3)$; passes through the point $(0, 5)$
16. Vertices: $(-2, 1)$, $(2, 1)$; passes through the point $(5, 4)$
17. Vertices: $(0, -3)$, $(4, -3)$; passes through the point $(-4, 5)$
18. Vertices: $(1, -3)$, $(1, -7)$; passes through the point $(5, -11)$

Sketching a Hyperbola **In Exercises 19–32, find the center, vertices, foci, and the equations of the asymptotes of the hyperbola. Then sketch the hyperbola using the asymptotes as an aid.**

19. $x^2 - y^2 = 1$
20. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$
21. $\dfrac{1}{36}y^2 - \dfrac{1}{100}x^2 = 1$
22. $\dfrac{1}{144}x^2 - \dfrac{1}{169}y^2 = 1$
23. $2y^2 - \dfrac{x^2}{2} = 2$
24. $\dfrac{y^2}{3} - 3x^2 = 3$
25. $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{1} = 1$
26. $\dfrac{(x+3)^2}{144} - \dfrac{(y-2)^2}{25} = 1$
27. $\dfrac{(y+6)^2}{1/9} - \dfrac{(x-2)^2}{1/4} = 1$
28. $\dfrac{(y-1)^2}{1/4} - \dfrac{(x+3)^2}{1/16} = 1$
29. $9x^2 - y^2 - 36x - 6y + 18 = 0$
30. $x^2 - 9y^2 + 36y - 72 = 0$
31. $4x^2 - y^2 + 8x + 2y - 1 = 0$
32. $16y^2 - x^2 + 2x + 64y + 64 = 0$

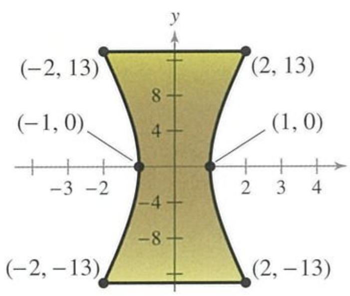
Graphing a Hyperbola **In Exercises 33–38, use a graphing utility to graph the hyperbola and its asymptotes. Find the center, vertices, and foci.**

33. $2x^2 - 3y^2 = 6$
34. $6y^2 - 3x^2 = 18$
35. $25y^2 - 9x^2 = 225$
36. $25x^2 - 4y^2 = 100$
37. $9y^2 - x^2 + 2x + 54y + 62 = 0$
38. $9x^2 - y^2 + 54x + 10y + 55 = 0$

Finding the Standard Equation of a Hyperbola In Exercises 39–48, find the standard form of the equation of the hyperbola with the given characteristics.

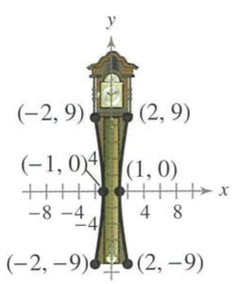
39. Vertices: $(\pm 1, 0)$; asymptotes: $y = \pm 5x$
40. Vertices: $(0, \pm 3)$; asymptotes: $y = \pm 3x$
41. Foci: $(0, \pm 8)$; asymptotes: $y = \pm 4x$
42. Foci: $(\pm 10, 0)$; asymptotes: $y = \pm \frac{3}{4}x$
43. Vertices: $(1, 2), (3, 2)$;
asymptotes: $y = x, y = 4 - x$
44. Vertices: $(3, 0), (3, 6)$;
asymptotes: $y = 6 - x, y = x$
45. Vertices: $(3, 0), (3, 4)$;
asymptotes: $y = \frac{2}{3}x, y = 4 - \frac{2}{3}x$
46. Vertices: $(-4, 1), (0, 1)$;
asymptotes: $y = x + 3, y = -x - 1$
47. Foci: $(-1, -1), (9, -1)$;
asymptotes: $y = \frac{3}{4}x - 4, y = -\frac{3}{4}x + 2$
48. Foci: $(9, \pm 2\sqrt{10})$;
asymptotes: $y = 3x - 27, y = -3x + 27$

49. **Art** A cross section of a sculpture can be modeled by a hyperbola (see figure).



(a) Write an equation that models the curved sides of the sculpture.
(b) Each unit in the coordinate plane represents 1 foot. Find the width of the sculpture at a height of 18 feet.

50. **Clock** The base of a clock has the shape of a hyperbola (see figure).



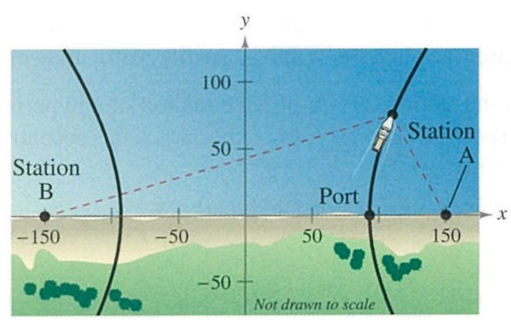
(a) Write an equation of the cross section of the base.
(b) Each unit in the coordinate plane represents $\frac{1}{2}$ foot. Find the width of the base 4 inches from the bottom.

51. **Sound Location** You and a friend live 4 miles apart. You hear a clap of thunder from lightning 18 seconds before your friend hears it. Where did the lightning occur? (Assume sound travels at 1100 feet per second.)

52. **Sound Location** Listening station A and listening station B are located at $(3300, 0)$ and $(-3300, 0)$, respectively. Station A detects an explosion 4 seconds before station B. (Assume the coordinate system is measured in feet and sound travels at 1100 feet per second.)
(a) Where did the explosion occur?
(b) Station C is located at $(3300, 1100)$ and detects the explosion 1 second after station A. Find the coordinates of the explosion.

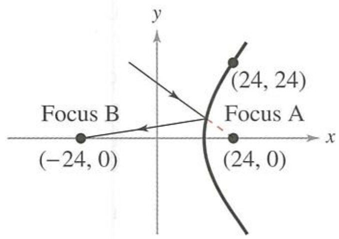
53. **Navigation**
Long-distance radio navigation for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci.

Assume that two stations 300 miles apart are positioned on a rectangular coordinate system with coordinates $(-150, 0)$ and $(150, 0)$ and that a ship is traveling on a hyperbolic path with coordinates $(x, 75)$ (see figure).



(a) Find the x-coordinate of the position of the ship when the time difference between the pulses from the transmitting stations is 1000 microseconds (0.001 second).
(b) Determine the distance between the port and station A.
(c) Find a linear equation that approximates the ship's path as it travels far away from the shore.

54. Hyperbolic Mirror A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at focus A is reflected to focus B (see figure). Find the vertex of the mirror when its mount at the top edge of the mirror has coordinates (24, 24).



Classifying a Conic from a General Equation **In Exercises 55–66, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.**

55. $9x^2 + 4y^2 - 18x + 16y - 119 = 0$
56. $x^2 + y^2 - 4x - 6y - 23 = 0$
57. $4x^2 - y^2 - 4x - 3 = 0$
58. $y^2 - 6y - 4x + 21 = 0$
59. $y^2 - 4x^2 + 4x - 2y - 4 = 0$
60. $y^2 + 12x + 4y + 28 = 0$
61. $4x^2 + 25y^2 + 16x + 250y + 541 = 0$
62. $4y^2 - 2x^2 - 4y - 8x - 15 = 0$
63. $25x^2 - 10x - 200y - 119 = 0$
64. $4y^2 + 4x^2 - 24x + 35 = 0$
65. $100x^2 + 100y^2 - 100x + 400y + 409 = 0$
66. $9x^2 + 4y^2 - 90x + 8y + 228 = 0$

Exploration

True or False? **In Exercises 67–69, determine whether the statement is true or false. Justify your answer.**

67. In the standard form of the equation of a hyperbola, the larger the ratio of b to a, the larger the eccentricity of the hyperbola.

68. If the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $a, b > 0$, intersect at right angles, then $a = b$.

69. The graph of

$$x^2 - y^2 + 4x - 4y = 0$$

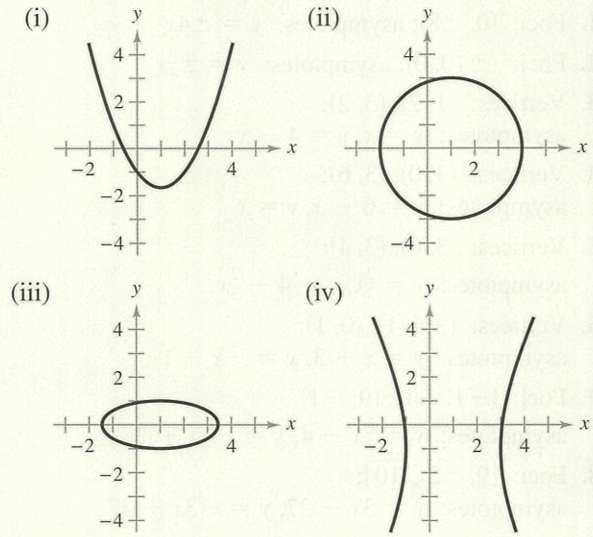
is a hyperbola.

70. **Think About It** Write an equation whose graph is the bottom half of the hyperbola

$$9x^2 - 54x - 4y^2 + 8y + 41 = 0.$$

71. **Writing** Explain how to use a rectangle to sketch the asymptotes of a hyperbola.

72. **HOW DO YOU SEE IT?** Match each equation with its graph.



(a) $4x^2 - y^2 - 8x - 2y - 13 = 0$
(b) $x^2 + y^2 - 2x - 8 = 0$
(c) $2x^2 - 4x - 3y - 3 = 0$
(d) $x^2 + 6y^2 - 2x - 5 = 0$

73. **Error Analysis** Describe the error in finding the asymptotes of the hyperbola

$$\frac{(y+5)^2}{9} - \frac{(x-3)^2}{4} = 1.$$

$$y = k \pm \frac{b}{a}(x - h)$$
$$= -5 \pm \tfrac{2}{3}(x - 3)$$

The asymptotes are
$y = \frac{2}{3}x - 7$ and $y = -\frac{2}{3}x - 3$.

74. **Think About It** Consider a hyperbola centered at the origin with a horizontal transverse axis. Use the definition of a hyperbola to derive its standard form.

75. **Points of Intersection** Sketch the circle $x^2 + y^2 = 4$. Then find the values of C so that the parabola $y = x^2 + C$ intersects the circle at the given number of points.

(a) 0 points
(b) 1 point
(c) 2 points
(d) 3 points
(e) 4 points