1.7 Transformations of Functions

Transformations of functions model many real-life applications. For example, in Exercise 61 on page 74, you will use a transformation of a function to model the number of horsepower required to overcome wind drag on an automobile.

- **Use vertical and horizontal shifts to sketch graphs of functions.**
- **Use reflections to sketch graphs of functions.**
- **Use nonrigid transformations to sketch graphs of functions.**

Shifting Graphs

Many functions have graphs that are transformations of the parent graphs summarized in Section 1.6. For example, you obtain the graph of

$$h(x) = x^2 + 2$$

by shifting the graph of $f(x) = x^2$ *up* two units, as shown in Figure 1.47. In function notation, h and f are related as follows.

$$h(x) = x^2 + 2 = f(x) + 2 \qquad \text{Upward shift of two units}$$

Similarly, you obtain the graph of

$$g(x) = (x - 2)^2$$

by shifting the graph of $f(x) = x^2$ to the *right* two units, as shown in Figure 1.48. In this case, the functions g and f have the following relationship.

$$g(x) = (x - 2)^2 = f(x - 2) \qquad \text{Right shift of two units}$$

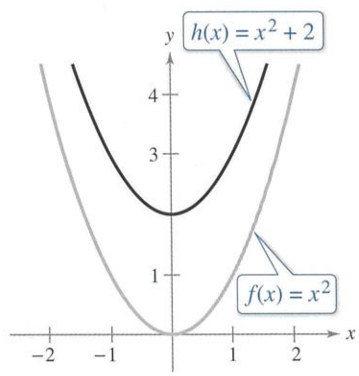


Figure 1.47

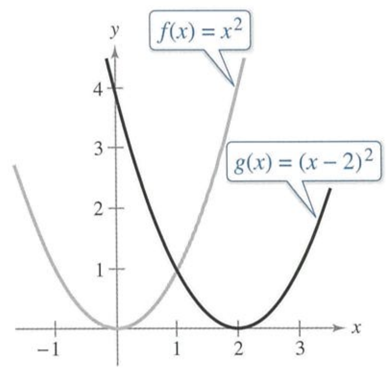


Figure 1.48

The list below summarizes this discussion about horizontal and vertical shifts.

REMARK In items 3 and 4, be sure you see that $h(x) = f(x - c)$ corresponds to a *right* shift and $h(x) = f(x + c)$ corresponds to a *left* shift for $c > 0$.

Vertical and Horizontal Shifts

Let c be a positive real number. **Vertical and horizontal shifts** in the graph of $y = f(x)$ are represented as follows.

1. Vertical shift c units *up:* $h(x) = f(x) + c$
2. Vertical shift c units *down:* $h(x) = f(x) - c$
3. Horizontal shift c units to the *right:* $h(x) = f(x - c)$
4. Horizontal shift c units to the *left:* $h(x) = f(x + c)$

Some graphs are obtained from combinations of vertical and horizontal shifts, as demonstrated in Example 1(b). Vertical and horizontal shifts generate a *family of functions,* each with the same shape but at a different location in the plane.

EXAMPLE 1 Shifting the Graph of a Function

Use the graph of $f(x) = x^3$ to sketch the graph of each function.

a. $g(x) = x^3 - 1$

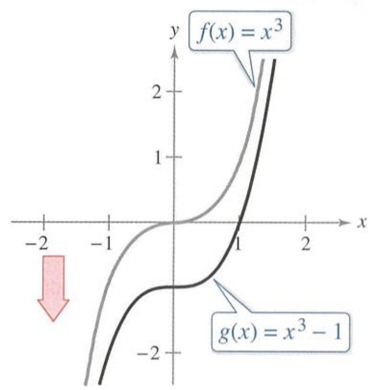
b. $h(x) = (x + 2)^3 + 1$

Solution

a. Relative to the graph of $f(x) = x^3$, the graph of

$$g(x) = x^3 - 1$$

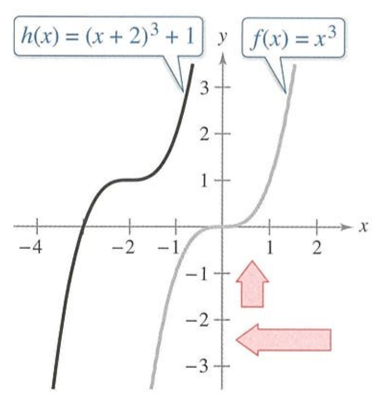
is a downward shift of one unit, as shown below.



b. Relative to the graph of $f(x) = x^3$, the graph of

$$h(x) = (x + 2)^3 + 1$$

is a left shift of two units and an upward shift of one unit, as shown below.



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Use the graph of $f(x) = x^3$ to sketch the graph of each function.

a. $h(x) = x^3 + 5$

b. $g(x) = (x - 3)^3 + 2$

In Example 1(a), note that $g(x) = f(x) - 1$ and in Example 1(b), $h(x) = f(x + 2) + 1$. In Example 1(b), you obtain the same result whether the vertical shift precedes the horizontal shift or the horizontal shift precedes the vertical shift.

Reflecting Graphs

Another common type of transformation is a **reflection.** For example, if you consider the x-axis to be a mirror, then the graph of $h(x) = -x^2$ is the mirror image (or reflection) of the graph of $f(x) = x^2$, as shown in Figure 1.49.

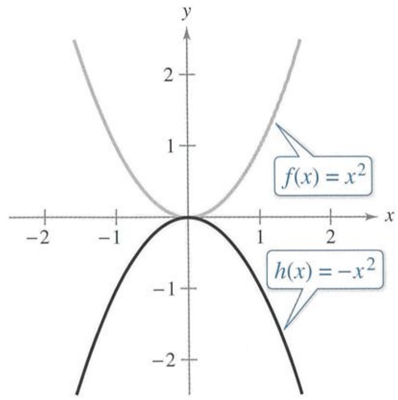


Figure 1.49

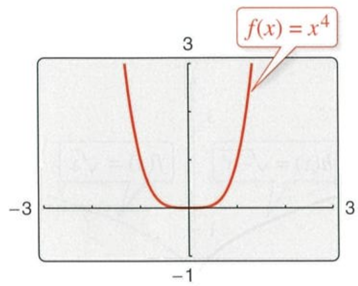


Figure 1.50

Reflections in the Coordinate Axes

Reflections in the coordinate axes of the graph of $y = f(x)$ are represented as follows.

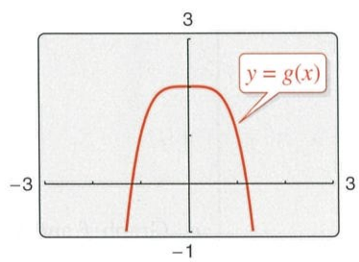
1. Reflection in the x-axis: $h(x) = -f(x)$
2. Reflection in the y-axis: $h(x) = f(-x)$

EXAMPLE 2 Writing Equations from Graphs

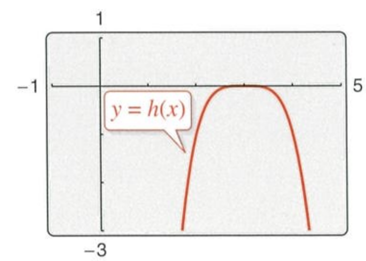
The graph of the function

$$f(x) = x^4$$

is shown in Figure 1.50. Each graph below is a transformation of the graph of f. Write an equation for the function represented by each graph.



(a)



(b)

Solution

a. The graph of g is a reflection in the x-axis *followed by* an upward shift of two units of the graph of $f(x) = x^4$. So, an equation for g is

$$g(x) = -x^4 + 2.$$

b. The graph of h is a right shift of three units *followed by* a reflection in the x-axis of the graph of $f(x) = x^4$. So, an equation for h is

$$h(x) = -(x - 3)^4.$$

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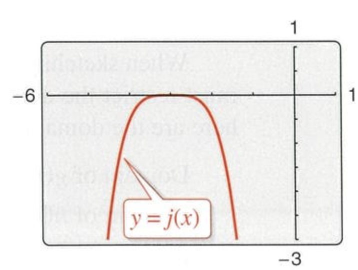
The graph is a transformation of the graph of $f(x) = x^4$. Write an equation for the function represented by the graph.



EXAMPLE 3 Reflections and Shifts

Compare the graph of each function with the graph of $f(x) = \sqrt{x}$.

a. $g(x) = -\sqrt{x}$ **b.** $h(x) = \sqrt{-x}$ **c.** $k(x) = -\sqrt{x+2}$

Algebraic Solution

a. The graph of g is a reflection of the graph of f in the x-axis because

$$g(x) = -\sqrt{x} = -f(x).$$

b. The graph of h is a reflection of the graph of f in the y-axis because

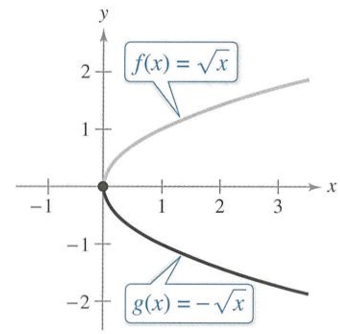
$$h(x) = \sqrt{-x} = f(-x).$$

c. The graph of k is a left shift of two units followed by a reflection in the x-axis because

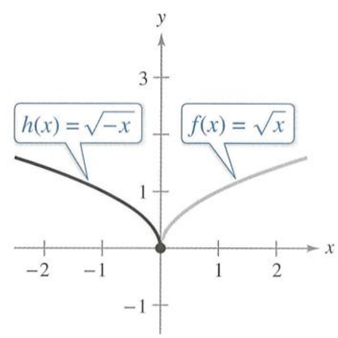
$$k(x) = -\sqrt{x+2} = -f(x+2).$$

Graphical Solution

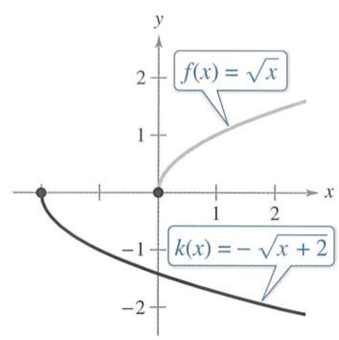
a. Graph f and g on the same set of coordinate axes. The graph of g is a reflection of the graph of f in the x-axis.



b. Graph f and h on the same set of coordinate axes. The graph of h is a reflection of the graph of f in the y-axis.



c. Graph f and k on the same set of coordinate axes. The graph of k is a left shift of two units followed by a reflection in the x-axis of the graph of f.



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Compare the graph of each function with the graph of

$$f(x) = \sqrt{x-1}.$$

a. $g(x) = -\sqrt{x-1}$ **b.** $h(x) = \sqrt{-x-1}$

When sketching the graphs of functions involving square roots, remember that you must restrict the domain to exclude negative numbers inside the radical. For instance, here are the domains of the functions in Example 3.

Domain of $g(x) = -\sqrt{x}$: $x \ge 0$

Domain of $h(x) = \sqrt{-x}$: $x \le 0$

Domain of $k(x) = -\sqrt{x+2}$: $x \ge -2$

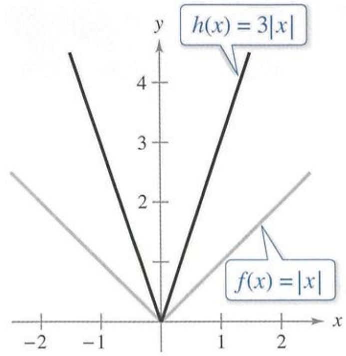

Figure 1.51

Nonrigid Transformations

Horizontal shifts, vertical shifts, and reflections are **rigid transformations** because the basic shape of the graph is unchanged. These transformations change only the *position* of the graph in the coordinate plane. **Nonrigid transformations** are those that cause a *distortion*—a change in the shape of the original graph. For example, a nonrigid transformation of the graph of $y = f(x)$ is represented by $g(x) = cf(x)$, where the transformation is a **vertical stretch** when $c > 1$ and a **vertical shrink** when $0 < c < 1$. Another nonrigid transformation of the graph of $y = f(x)$ is represented by $h(x) = f(cx)$, where the transformation is a **horizontal shrink** when $c > 1$ and a **horizontal stretch** when $0 < c < 1$.

EXAMPLE 4 Nonrigid Transformations

Compare the graph of each function with the graph of $f(x) = |x|$.

a. $h(x) = 3|x|$ **b.** $g(x) = \frac{1}{3}|x|$

Solution

a. Relative to the graph of $f(x) = |x|$, the graph of $h(x) = 3|x| = 3f(x)$ is a vertical stretch (each y-value is multiplied by 3). (See Figure 1.51.)

b. Similarly, the graph of $g(x) = \frac{1}{3}|x| = \frac{1}{3}f(x)$ is a vertical shrink (each y-value is multiplied by $\frac{1}{3}$) of the graph of f. (See Figure 1.52.)

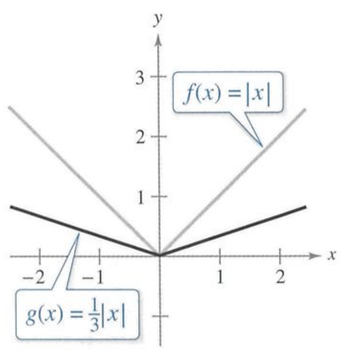

Figure 1.52

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Compare the graph of each function with the graph of $f(x) = x^2$.

a. $g(x) = 4x^2$ **b.** $h(x) = \frac{1}{4}x^2$

EXAMPLE 5 Nonrigid Transformations

See LarsonPrecalculus.com for an interactive version of this type of example.

Compare the graph of each function with the graph of $f(x) = 2 - x^3$.

a. $g(x) = f(2x)$ **b.** $h(x) = f\left(\frac{1}{2}x\right)$

Solution

a. Relative to the graph of $f(x) = 2 - x^3$, the graph of $g(x) = f(2x) = 2 - (2x)^3 = 2 - 8x^3$ is a horizontal shrink ($c > 1$). (See Figure 1.53.)

b. Similarly, the graph of $h(x) = f\left(\frac{1}{2}x\right) = 2 - \left(\frac{1}{2}x\right)^3 = 2 - \frac{1}{8}x^3$ is a horizontal stretch ($0 < c < 1$) of the graph of f. (See Figure 1.54.)

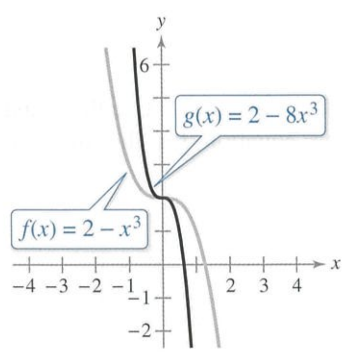

Figure 1.53

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Compare the graph of each function with the graph of $f(x) = x^2 + 3$.

a. $g(x) = f(2x)$ **b.** $h(x) = f\left(\frac{1}{2}x\right)$

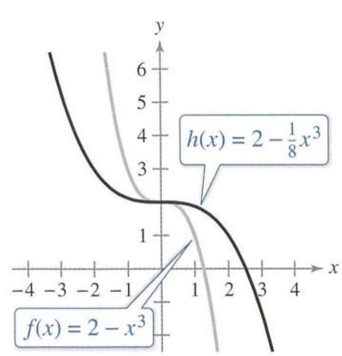


Figure 1.54

Summarize (Section 1.7)

1. Explain how to shift the graph of a function vertically and horizontally *(page 67)*. For an example of shifting the graph of a function, see Example 1.
2. Explain how to reflect the graph of a function in the x-axis and in the y-axis *(page 69)*. For examples of reflecting graphs of functions, see Examples 2 and 3.
3. Describe nonrigid transformations of the graph of a function *(page 71)*. For examples of nonrigid transformations, see Examples 4 and 5.

1.7 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary

In Exercises 1–3, fill in the blanks.

1. Horizontal shifts, vertical shifts, and reflections are ________ transformations.
2. A reflection in the x-axis of the graph of $y = f(x)$ is represented by $h(x) =$ ________, while a reflection in the y-axis of the graph of $y = f(x)$ is represented by $h(x) =$ ________.
3. A nonrigid transformation of the graph of $y = f(x)$ represented by $g(x) = cf(x)$ is a ________ ________ when $c > 1$ and a ________ ________ when $0 < c < 1$.
4. Match each function h with the transformation it represents, where $c > 0$.

(a) $h(x) = f(x) + c$ (i) A horizontal shift of f, c units to the right
(b) $h(x) = f(x) - c$ (ii) A vertical shift of f, c units down
(c) $h(x) = f(x + c)$ (iii) A horizontal shift of f, c units to the left
(d) $h(x) = f(x - c)$ (iv) A vertical shift of f, c units up

Skills and Applications

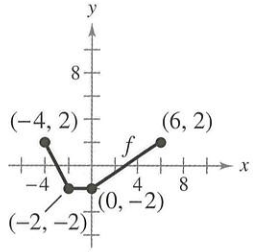
5. **Shifting the Graph of a Function** For each function, sketch the graphs of the function when $c = -2$, -1, 1, and 2 on the same set of coordinate axes.
(a) $f(x) = |x| + c$ (b) $f(x) = |x - c|$
6. **Shifting the Graph of a Function** For each function, sketch the graphs of the function when $c = -3$, -2, 2, and 3 on the same set of coordinate axes.
(a) $f(x) = \sqrt{x} + c$ (b) $f(x) = \sqrt{x - c}$
7. **Shifting the Graph of a Function** For each function, sketch the graphs of the function when $c = -4$, -1, 2, and 5 on the same set of coordinate axes.
(a) $f(x) = [\![x]\!] + c$ (b) $f(x) = [\![x + c]\!]$
8. **Shifting the Graph of a Function** For each function, sketch the graphs of the function when $c = -3$, -2, 1, and 2 on the same set of coordinate axes.

(a) $f(x) = \begin{cases} x^2 + c, & x < 0 \\ -x^2 + c, & x \ge 0 \end{cases}$

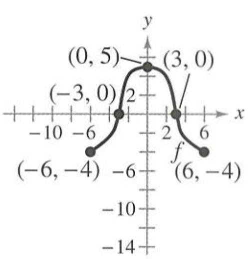
(b) $f(x) = \begin{cases} (x + c)^2, & x < 0 \\ -(x + c)^2, & x \ge 0 \end{cases}$

Sketching Transformations **In Exercises 9 and 10, use the graph of f to sketch each graph. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

9. (a) $y = f(-x)$
(b) $y = f(x) + 4$
(c) $y = 2f(x)$
(d) $y = -f(x - 4)$
(e) $y = f(x) - 3$
(f) $y = -f(x) - 1$
(g) $y = f(2x)$

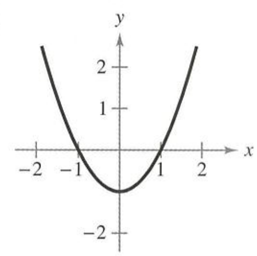


10. (a) $y = f(x - 5)$
(b) $y = -f(x) + 3$
(c) $y = \frac{1}{3}f(x)$
(d) $y = -f(x + 1)$
(e) $y = f(-x)$
(f) $y = f(x) - 10$
(g) $y = f\left(\frac{1}{3}x\right)$

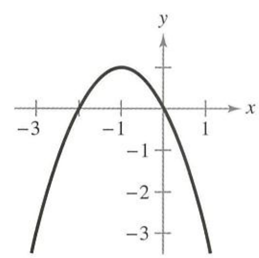


11. **Writing Equations from Graphs** Use the graph of $f(x) = x^2$ to write an equation for the function represented by each graph.

(a)

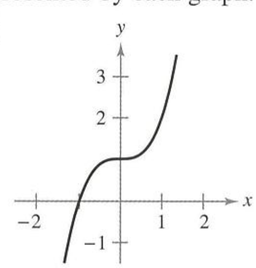


(b)

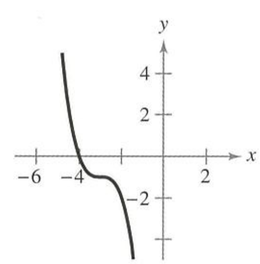


12. **Writing Equations from Graphs** Use the graph of $f(x) = x^3$ to write an equation for the function represented by each graph.

(a)

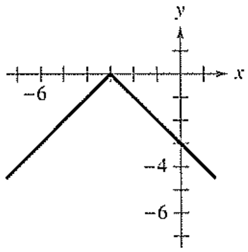


(b)

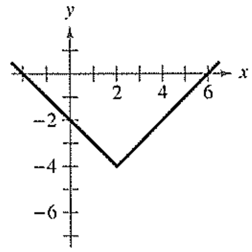


13. Writing Equations from Graphs Use the graph of $f(x) = |x|$ to write an equation for the function represented by each graph.

(a)

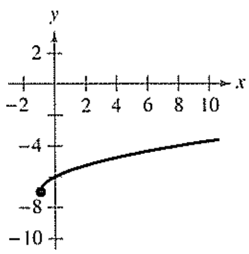


(b)

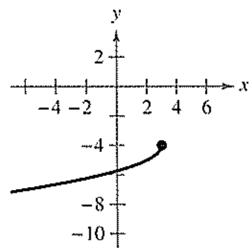


14. Writing Equations from Graphs Use the graph of $f(x) = \sqrt{x}$ to write an equation for the function represented by each graph.

(a)

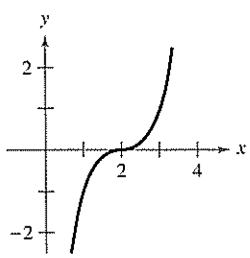


(b)

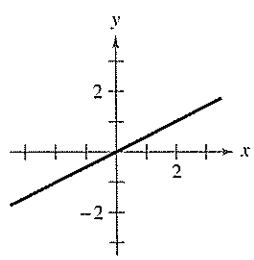


Writing Equations from Graphs **In Exercises 15–20, identify the parent function and the transformation represented by the graph. Write an equation for the function represented by the graph.**

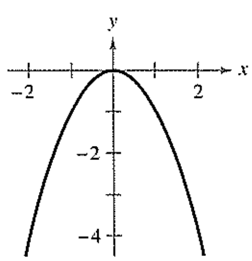
15.



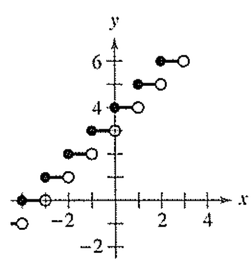
16.



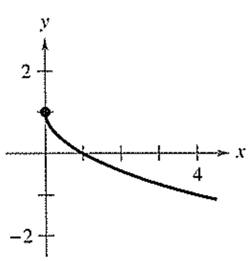
17.



18.



19.



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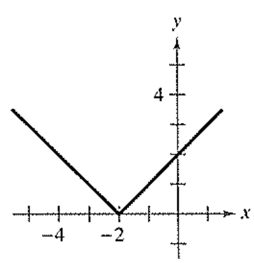


Describing Transformations **In Exercises 21–38, g is related to one of the parent functions described in Section 1.6. (a) Identify the parent function f. (b) Describe the sequence of transformations from f to g. (c) Sketch the graph of g. (d) Use function notation to write g in terms of f.**

21. $g(x) = x^2 + 6$

22. $g(x) = x^2 - 2$

23. $g(x) = -(x - 2)^3$

24. $g(x) = -(x + 1)^3$

25. $g(x) = -3 - (x + 1)^2$

26. $g(x) = 4 - (x - 2)^2$

27. $g(x) = |x - 1| + 2$

28. $g(x) = |x + 3| - 2$

29. $g(x) = 2\sqrt{x}$

30. $g(x) = \frac{1}{2}\sqrt{x}$

31. $g(x) = 2[\![x]\!] - 1$

32. $g(x) = -[\![x]\!] + 1$

33. $g(x) = |2x|$

34. $g(x) = \left|\frac{1}{2}x\right|$

35. $g(x) = -2x^2 + 1$

36. $g(x) = \frac{1}{2}x^2 - 2$

37. $g(x) = 3|x - 1| + 2$

38. $g(x) = -2|x + 1| - 3$

Writing an Equation from a Description **In Exercises 39–46, write an equation for the function whose graph is described.**

39. The shape of $f(x) = x^2$, but shifted three units to the right and seven units down

40. The shape of $f(x) = x^2$, but shifted two units to the left, nine units up, and then reflected in the x-axis

41. The shape of $f(x) = x^3$, but shifted 13 units to the right

42. The shape of $f(x) = x^3$, but shifted six units to the left, six units down, and then reflected in the y-axis

43. The shape of $f(x) = |x|$, but shifted 12 units up and then reflected in the x-axis

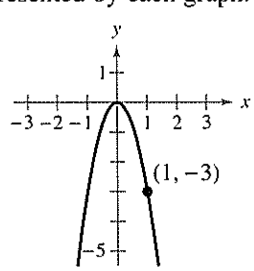
44. The shape of $f(x) = |x|$, but shifted four units to the left and eight units down

45. The shape of $f(x) = \sqrt{x}$, but shifted six units to the left and then reflected in both the x-axis and the y-axis

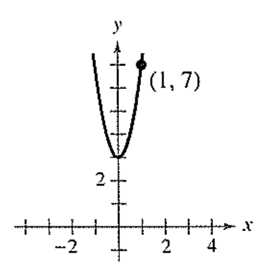
46. The shape of $f(x) = \sqrt{x}$, but shifted nine units down and then reflected in both the x-axis and the y-axis

47. Writing Equations from Graphs Use the graph of $f(x) = x^2$ to write an equation for the function represented by each graph.

(a)



(b)

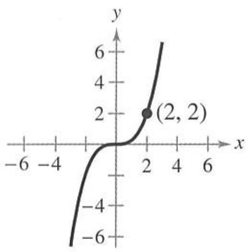


48. Writing Equations from Graphs Use the graph of

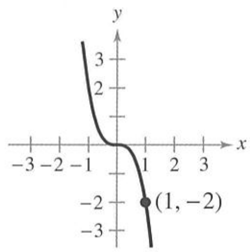
$f(x) = x^3$

to write an equation for the function represented by each graph.

(a)



(b)

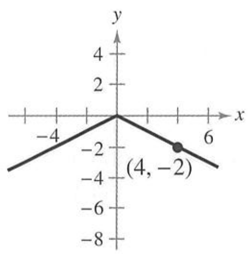


49. Writing Equations from Graphs Use the graph of

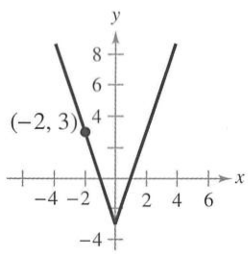
$f(x) = |x|$

to write an equation for the function represented by each graph.

(a)



(b)

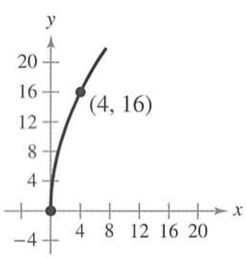


50. Writing Equations from Graphs Use the graph of

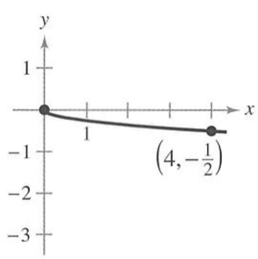
$f(x) = \sqrt{x}$

to write an equation for the function represented by each graph.

(a)

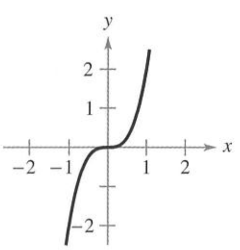


(b)

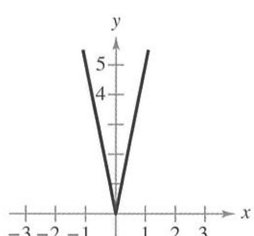


Writing Equations from Graphs **In Exercises 51–56, identify the parent function and the transformation represented by the graph. Write an equation for the function represented by the graph. Then use a graphing utility to verify your answer.**

51.



52.



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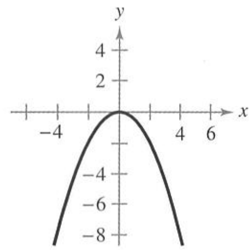

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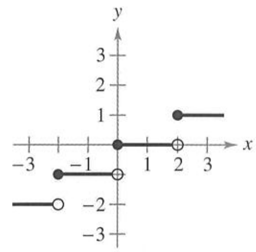

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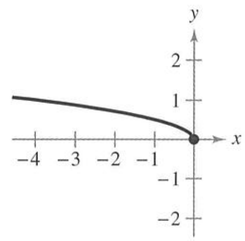
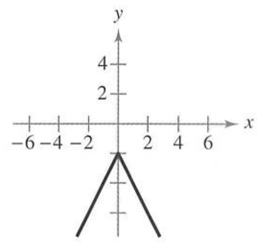
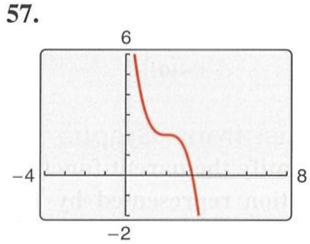


56.

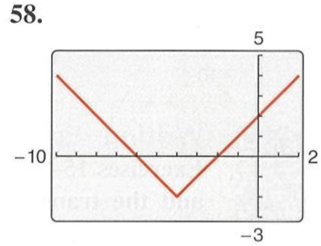


Writing Equations from Graphs **In Exercises 57–60, write an equation for the transformation of the parent function.**

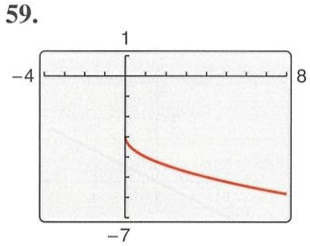
57.



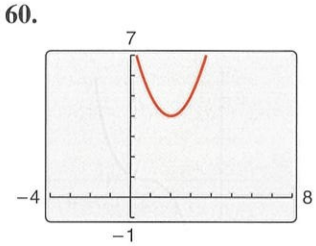
58.



59.



60.



61. Automobile Aerodynamics

The horsepower H required to overcome wind drag on a particular automobile is given by

$H(x) = 0.00004636x^3$

where x is the speed of the car (in miles per hour).

(a) Use a graphing utility to graph the function.

(b) Rewrite the horsepower function so that x represents the speed in kilometers per hour. [Find $H(x/1.6)$.] Identify the type of transformation applied to the graph of the horsepower function.

62. Households The number N (in millions) of households in the United States from 2000 through 2014 can be approximated by

$$N(x) = -0.023(x - 33.12)^2 + 131, \quad 0 \le t \le 14$$

where t represents the year, with $t = 0$ corresponding to 2000. *(Source: U.S. Census Bureau)*

(a) Describe the transformation of the parent function $f(x) = x^2$. Then use a graphing utility to graph the function over the specified domain.

(b) Find the average rate of change of the function from 2000 to 2014. Interpret your answer in the context of the problem.

(c) Use the model to predict the number of households in the United States in 2022. Does your answer seem reasonable? Explain.

Exploration

True or False? **In Exercises 63–66, determine whether the statement is true or false. Justify your answer.**

63. The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

64. The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

65. The graphs of $f(x) = |x| + 6$ and $f(x) = |-x| + 6$ are identical.

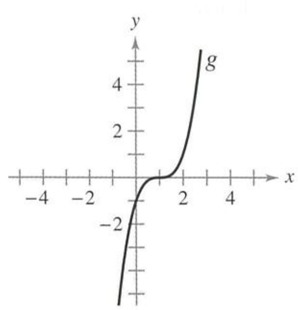
66. If the graph of the parent function $f(x) = x^2$ is shifted six units to the right, three units up, and reflected in the x-axis, then the point $(-2, 19)$ will lie on the graph of the transformation.

67. Finding Points on a Graph The graph of $y = f(x)$ passes through the points $(0, 1)$, $(1, 2)$, and $(2, 3)$. Find the corresponding points on the graph of $y = f(x + 2) - 1$.

68. Think About It Two methods of graphing a function are plotting points and translating a parent function as shown in this section. Which method of graphing do you prefer to use for each function? Explain.

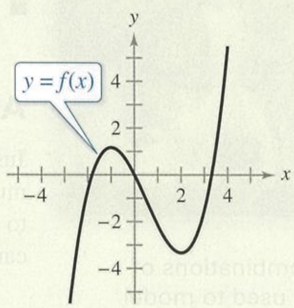
(a) $f(x) = 3x^2 - 4x + 1$ (b) $f(x) = 2(x - 1)^2 - 6$

69. Error Analysis Describe the error.



The graph of g is a right shift of one unit of the graph of $f(x) = x^3$. So, an equation for g is $g(x) = (x + 1)^3$.

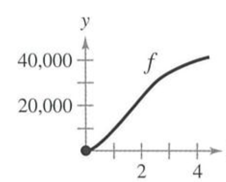
70. HOW DO YOU SEE IT? Use the graph of $y = f(x)$ to find the open intervals on which the graph of each transformation is increasing and decreasing. If not possible, state the reason.



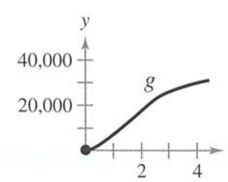
(a) $y = f(-x)$ (b) $y = -f(x)$ (c) $y = \frac{1}{2}f(x)$

(d) $y = -f(x - 1)$ (e) $y = f(x - 2) + 1$

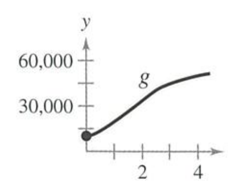
71. Describing Profits Management originally predicted that the profits from the sales of a new product could be approximated by the graph of the function f shown. The actual profits are represented by the graph of the function g along with a verbal description. Use the concepts of transformations of graphs to write g in terms of f.



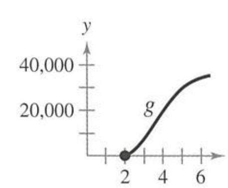
(a) The profits were only three-fourths as large as expected.



(b) The profits were consistently $10,000 greater than predicted.



(c) There was a two-year delay in the introduction of the product. After sales began, profits grew as expected.



72. Reversing the Order of Transformations Reverse the order of transformations in Example 2(a). Do you obtain the same graph? Do the same for Example 2(b). Do you obtain the same graph? Explain.

1.8 Combinations of Functions: Composite Functions

Arithmetic combinations of functions are used to model and solve real-life problems. For example, in Exercise 60 on page 82, you will use arithmetic combinations of functions to analyze numbers of pets in the United States.

- **Add, subtract, multiply, and divide functions.**
- **Find the composition of one function with another function.**
- **Use combinations and compositions of functions to model and solve real-life problems.**

Arithmetic Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two *functions* can be combined to create new functions. For example, the functions $f(x) = 2x - 3$ and $g(x) = x^2 - 1$ can be combined to form the sum, difference, product, and quotient of f and g.

$$f(x) + g(x) = (2x - 3) + (x^2 - 1) = x^2 + 2x - 4 \quad \text{Sum}$$

$$f(x) - g(x) = (2x - 3) - (x^2 - 1) = -x^2 + 2x - 2 \quad \text{Difference}$$

$$f(x)g(x) = (2x - 3)(x^2 - 1) = 2x^3 - 3x^2 - 2x + 3 \quad \text{Product}$$

$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 - 1}, \quad x \neq \pm 1 \quad \text{Quotient}$$

The domain of an **arithmetic combination** of functions f and g consists of all real numbers that are common to the domains of f and g. In the case of the quotient $f(x)/g(x)$, there is the further restriction that $g(x) \neq 0$.

Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the *sum, difference, product,* and *quotient* of f and g are defined as follows.

1. **Sum:** $(f + g)(x) = f(x) + g(x)$
2. **Difference:** $(f - g)(x) = f(x) - g(x)$
3. **Product:** $(fg)(x) = f(x) \cdot g(x)$
4. **Quotient:** $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

EXAMPLE 1 Finding the Sum of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f + g)(x)$. Then evaluate the sum when $x = 3$.

Solution The sum of f and g is

$$(f + g)(x) = f(x) + g(x) = (2x + 1) + (x^2 + 2x - 1) = x^2 + 4x.$$

When $x = 3$, the value of this sum is

$$(f + g)(3) = 3^2 + 4(3) = 21.$$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Given $f(x) = x^2$ and $g(x) = 1 - x$, find $(f + g)(x)$. Then evaluate the sum when $x = 2$.

EXAMPLE 2 Finding the Difference of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f - g)(x)$. Then evaluate the difference when $x = 2$.

Solution The difference of f and g is

$$(f - g)(x) = f(x) - g(x) = (2x + 1) - (x^2 + 2x - 1) = -x^2 + 2.$$

When $x = 2$, the value of this difference is

$$(f - g)(2) = -(2)^2 + 2 = -2.$$

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Given $f(x) = x^2$ and $g(x) = 1 - x$, find $(f - g)(x)$. Then evaluate the difference when $x = 3$.

EXAMPLE 3 Finding the Product of Two Functions

Given $f(x) = x^2$ and $g(x) = x - 3$, find $(fg)(x)$. Then evaluate the product when $x = 4$.

Solution The product of f and g is

$$(fg)(x) = f(x)g(x) = (x^2)(x - 3) = x^3 - 3x^2.$$

When $x = 4$, the value of this product is

$$(fg)(4) = 4^3 - 3(4)^2 = 16.$$

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Given $f(x) = x^2$ and $g(x) = 1 - x$, find $(fg)(x)$. Then evaluate the product when $x = 3$.

In Examples 1–3, both f and g have domains that consist of all real numbers. So, the domains of $f + g$, $f - g$, and fg are also the set of all real numbers. Remember to consider any restrictions on the domains of f and g when forming the sum, difference, product, or quotient of f and g.

EXAMPLE 4 Finding the Quotients of Two Functions

Find $(f/g)(x)$ and $(g/f)(x)$ for the functions $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$. Then find the domains of f/g and g/f.

Solution The quotient of f and g is

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4 - x^2}}$$

and the quotient of g and f is

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4 - x^2}}{\sqrt{x}}.$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these domains is $[0, 2]$. So, the domains of f/g and g/f are as follows.

Domain of f/g: $[0, 2)$ Domain of g/f: $(0, 2]$

REMARK Note that the domain of f/g includes $x = 0$, but not $x = 2$, because $x = 2$ yields a zero in the denominator, whereas the domain of g/f includes $x = 2$, but not $x = 0$, because $x = 0$ yields a zero in the denominator.

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Find $(f/g)(x)$ and $(g/f)(x)$ for the functions $f(x) = \sqrt{x - 3}$ and $g(x) = \sqrt{16 - x^2}$. Then find the domains of f/g and g/f.

Composition of Functions

Another way of combining two functions is to form the **composition** of one with the other. For example, if $f(x) = x^2$ and $g(x) = x + 1$, then the composition of f with g is

$$f(g(x)) = f(x + 1)$$
$$= (x + 1)^2.$$

This composition is denoted as $f \circ g$ and reads as "f composed with g."

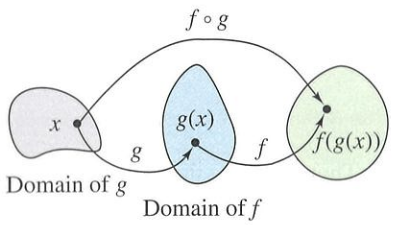

Figure 1.55

Definition of Composition of Two Functions

The **composition** of the function f with the function g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 1.55.)

EXAMPLE 5 Compositions of Functions

See LarsonPrecalculus.com for an interactive version of this type of example.

Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, find the following.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$ **c.** $(g \circ f)(-2)$

Solution

a. The composition of f with g is as shown.

$(f \circ g)(x) = f(g(x))$ Definition of $f \circ g$

$= f(4 - x^2)$ Definition of $g(x)$

$= (4 - x^2) + 2$ Definition of $f(x)$

$= -x^2 + 6$ Simplify.

b. The composition of g with f is as shown.

$(g \circ f)(x) = g(f(x))$ Definition of $g \circ f$

$= g(x + 2)$ Definition of $f(x)$

$= 4 - (x + 2)^2$ Definition of $g(x)$

$= 4 - (x^2 + 4x + 4)$ Expand.

$= -x^2 - 4x$ Simplify.

Note that, in this case, $(f \circ g)(x) \neq (g \circ f)(x)$.

c. Evaluate the result of part (b) when $x = -2$.

$(g \circ f)(-2) = -(-2)^2 - 4(-2)$ Substitute.

$= -4 + 8$ Simplify.

$= 4$ Simplify.

REMARK The tables of values below help illustrate the composition $(f \circ g)(x)$ in Example 5(a).

x	0	1	2	3
$g(x)$	4	3	0	−5

$g(x)$	4	3	0	−5
$f(g(x))$	6	5	2	−3

x	0	1	2	3
$f(g(x))$	6	5	2	−3

Note that the first two tables are combined (or "composed") to produce the values in the third table.

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Given $f(x) = 2x + 5$ and $g(x) = 4x^2 + 1$, find the following.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$ **c.** $(f \circ g)\left(-\frac{1}{2}\right)$

EXAMPLE 6 Finding the Domain of a Composite Function

Find the domain of $f \circ g$ for the functions

$$f(x) = x^2 - 9 \quad \text{and} \quad g(x) = \sqrt{9 - x^2}.$$

Algebraic Solution

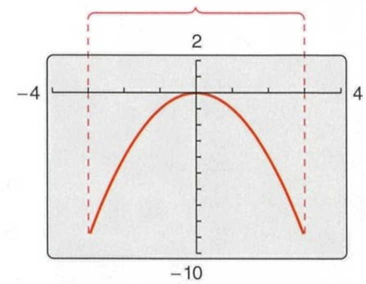
Find the composition of the functions.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\sqrt{9 - x^2}\right) \\ &= \left(\sqrt{9 - x^2}\right)^2 - 9 \\ &= 9 - x^2 - 9 \\ &= -x^2\end{aligned}$$

The domain of $f \circ g$ is restricted to the x-values in the domain of g for which $g(x)$ is in the domain of f. The domain of $f(x) = x^2 - 9$ is the set of all real numbers, which includes all real values of g. So, the domain of $f \circ g$ is the entire domain of $g(x) = \sqrt{9 - x^2}$, which is $[-3, 3]$.

Graphical Solution

Use a graphing utility to graph $f \circ g$.



From the graph, you can determine that the domain of $f \circ g$ is $[-3, 3]$.

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Find the domain of $f \circ g$ for the functions $f(x) = \sqrt{x}$ and $g(x) = x^2 + 4$.

In Examples 5 and 6, you formed the composition of two given functions. In calculus, it is also important to be able to identify two functions that make up a given composite function. For example, the function $h(x) = (3x - 5)^3$ is the composition of $f(x) = x^3$ and $g(x) = 3x - 5$. That is,

$$h(x) = (3x - 5)^3 = [g(x)]^3 = f(g(x)).$$

Basically, to "decompose" a composite function, look for an "inner" function and an "outer" function. In the function h above, $g(x) = 3x - 5$ is the inner function and $f(x) = x^3$ is the outer function.

EXAMPLE 7 Decomposing a Composite Function

Write the function $h(x) = \dfrac{1}{(x - 2)^2}$ as a composition of two functions.

Solution Consider $g(x) = x - 2$ as the inner function and $f(x) = \dfrac{1}{x^2} = x^{-2}$ as the outer function. Then write

$$\begin{aligned}h(x) &= \frac{1}{(x - 2)^2} \\ &= (x - 2)^{-2} \\ &= f(x - 2) \\ &= f(g(x)).\end{aligned}$$

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Write the function $h(x) = \dfrac{\sqrt[3]{8 - x}}{5}$ as a composition of two functions.

Application

EXAMPLE 8 **Bacteria Count**

Refrigerated foods can have two types of bacteria: pathogenic bacteria, which can cause foodborne illness, and spoilage bacteria, which give foods an unpleasant look, smell, taste, or texture.

The number N of bacteria in a refrigerated food is given by

$$N(T) = 20T^2 - 80T + 500, \quad 2 \le T \le 14$$

where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature of the food is given by

$$T(t) = 4t + 2, \quad 0 \le t \le 3$$

where t is the time in hours.

a. Find and interpret $(N \circ T)(t)$.

b. Find the time when the bacteria count reaches 2000.

Solution

a.
$$\begin{aligned}(N \circ T)(t) &= N(T(t)) \\ &= 20(4t + 2)^2 - 80(4t + 2) + 500 \\ &= 20(16t^2 + 16t + 4) - 320t - 160 + 500 \\ &= 320t^2 + 320t + 80 - 320t - 160 + 500 \\ &= 320t^2 + 420\end{aligned}$$

The composite function $N \circ T$ represents the number of bacteria in the food as a function of the amount of time the food has been out of refrigeration.

b. The bacteria count reaches 2000 when $320t^2 + 420 = 2000$. By solving this equation algebraically, you find that the count reaches 2000 when $t \approx 2.2$ hours. Note that the negative solution $t \approx -2.2$ hours is rejected because it is not in the domain of the composite function.

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The number N of bacteria in a refrigerated food is given by

$$N(T) = 8T^2 - 14T + 200, \quad 2 \le T \le 12$$

where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature of the food is given by

$$T(t) = 2t + 2, \quad 0 \le t \le 5$$

where t is the time in hours.

a. Find $(N \circ T)(t)$.

b. Find the time when the bacteria count reaches 1000.

Summarize (Section 1.8)

1. Explain how to add, subtract, multiply, and divide functions *(page 76)*. For examples of finding arithmetic combinations of functions, see Examples 1–4.
2. Explain how to find the composition of one function with another function *(page 78)*. For examples that use compositions of functions, see Examples 5–7.
3. Describe a real-life example that uses a composition of functions *(page 80, Example 8)*.

1.8 Exercises

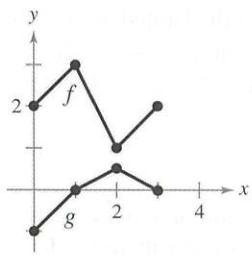
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: **Fill in the blanks.**

1. Two functions f and g can be combined by the arithmetic operations of ________, ________, ________, and ________ to create new functions.
2. The ________ of the function f with the function g is $(f \circ g)(x) = f(g(x))$.

Skills and Applications

Graphing the Sum of Two Functions **In Exercises 3 and 4, use the graphs of f and g to graph $h(x) = (f + g)(x)$. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

3.


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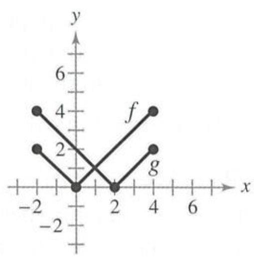


Finding Arithmetic Combinations of Functions **In Exercises 5–12, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?**

5. $f(x) = x + 2, \quad g(x) = x - 2$
6. $f(x) = 2x - 5, \quad g(x) = 2 - x$
7. $f(x) = x^2, \quad g(x) = 4x - 5$
8. $f(x) = 3x + 1, \quad g(x) = x^2 - 16$
9. $f(x) = x^2 + 6, \quad g(x) = \sqrt{1 - x}$
10. $f(x) = \sqrt{x^2 - 4}, \quad g(x) = \dfrac{x^2}{x^2 + 1}$
11. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = x^3$
12. $f(x) = \dfrac{2}{x}, \quad g(x) = \dfrac{1}{x^2 - 1}$

Evaluating an Arithmetic Combination of Functions **In Exercises 13–24, evaluate the function for $f(x) = x + 3$ and $g(x) = x^2 - 2$.**

13. $(f + g)(2)$
14. $(f + g)(-1)$
15. $(f - g)(0)$
16. $(f - g)(1)$
17. $(f - g)(3t)$
18. $(f + g)(t - 2)$
19. $(fg)(6)$
20. $(fg)(-6)$
21. $(f/g)(5)$
22. $(f/g)(0)$
23. $(f/g)(-1) - g(3)$
24. $(fg)(5) + f(4)$

Graphical Reasoning **In Exercises 25–28, use a graphing utility to graph f, g, and $f + g$ in the same viewing window. Which function contributes most to the magnitude of the sum when $0 \le x \le 2$? Which function contributes most to the magnitude of the sum when $x > 6$?**

25. $f(x) = 3x, \quad g(x) = -\dfrac{x^3}{10}$
26. $f(x) = \dfrac{x}{2}, \quad g(x) = \sqrt{x}$
27. $f(x) = 3x + 2, \quad g(x) = -\sqrt{x + 5}$
28. $f(x) = x^2 - \frac{1}{2}, \quad g(x) = -3x^2 - 1$

Finding Compositions of Functions **In Exercises 29–34, find (a) $f \circ g$, (b) $g \circ f$, and (c) $g \circ g$.**

29. $f(x) = x + 8, \quad g(x) = x - 3$
30. $f(x) = -4x, \quad g(x) = x + 7$
31. $f(x) = x^2, \quad g(x) = x - 1$
32. $f(x) = 3x, \quad g(x) = x^4$
33. $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x^3 + 1$
34. $f(x) = x^3, \quad g(x) = \dfrac{1}{x}$

Finding Domains of Functions and Composite Functions **In Exercises 35–42, find (a) $f \circ g$ and (b) $g \circ f$. Find the domain of each function and of each composite function.**

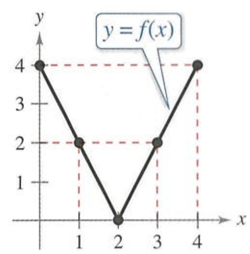
35. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2$
36. $f(x) = \sqrt[3]{x - 5}, \quad g(x) = x^3 + 1$
37. $f(x) = x^3, \quad g(x) = x^{2/3}$
38. $f(x) = x^5, \quad g(x) = \sqrt[4]{x}$
39. $f(x) = |x|, \quad g(x) = x + 6$
40. $f(x) = |x - 4|, \quad g(x) = 3 - x$
41. $f(x) = \dfrac{1}{x}, \quad g(x) = x + 3$
42. $f(x) = \dfrac{3}{x^2 - 1}, \quad g(x) = x + 1$

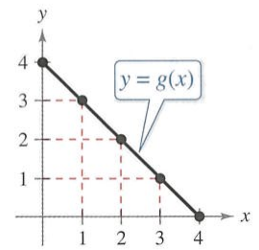
Graphing Combinations of Functions In Exercises 43 and 44, on the same set of coordinate axes, (a) graph the functions f, g, and $f + g$ and (b) graph the functions f, g, and $f \circ g$.

43. $f(x) = \frac{1}{2}x, \quad g(x) = x - 4$

44. $f(x) = x + 3, \quad g(x) = x^2$

Evaluating Combinations of Functions In Exercises 45–48, use the graphs of f and g to evaluate the functions.





45. (a) $(f + g)(3)$ (b) $(f/g)(2)$

46. (a) $(f - g)(1)$ (b) $(fg)(4)$

47. (a) $(f \circ g)(2)$ (b) $(g \circ f)(2)$

48. (a) $(f \circ g)(1)$ (b) $(g \circ f)(3)$

Decomposing a Composite Function In Exercises 49–56, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)

49. $h(x) = (2x + 1)^2$ **50.** $h(x) = (1 - x)^3$

51. $h(x) = \sqrt[3]{x^2 - 4}$ **52.** $h(x) = \sqrt{9 - x}$

53. $h(x) = \dfrac{1}{x + 2}$ **54.** $h(x) = \dfrac{4}{(5x + 2)^2}$

55. $h(x) = \dfrac{-x^2 + 3}{4 - x^2}$

56. $h(x) = \dfrac{27x^3 + 6x}{10 - 27x^3}$

57. Stopping Distance The research and development department of an automobile manufacturer determines that when a driver is required to stop quickly to avoid an accident, the distance (in feet) the car travels during the driver's reaction time is given by $R(x) = \frac{3}{4}x$, where x is the speed of the car in miles per hour. The distance (in feet) the car travels while the driver is braking is given by $B(x) = \frac{1}{15}x^2$.

(a) Find the function that represents the total stopping distance T.

(b) Graph the functions R, B, and T on the same set of coordinate axes for $0 \le x \le 60$.

(c) Which function contributes most to the magnitude of the sum at higher speeds? Explain.

58. Business The annual cost C (in thousands of dollars) and revenue R (in thousands of dollars) for a company each year from 2010 through 2016 can be approximated by the models

$$C = 254 - 9t + 1.1t^2 \quad \text{and} \quad R = 341 + 3.2t$$

where t is the year, with $t = 10$ corresponding to 2010.

(a) Write a function P that represents the annual profit of the company.

(b) Use a graphing utility to graph C, R, and P in the same viewing window.

59. Vital Statistics Let $b(t)$ be the number of births in the United States in year t, and let $d(t)$ represent the number of deaths in the United States in year t, where $t = 10$ corresponds to 2010.

(a) If $p(t)$ is the population of the United States in year t, find the function $c(t)$ that represents the percent change in the population of the United States.

(b) Interpret $c(16)$.

60. Pets

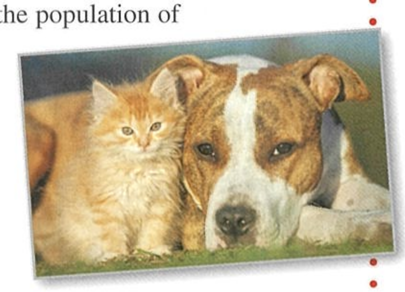
Let $d(t)$ be the number of dogs in the United States in year t, and let $c(t)$ be the number of cats in the United States in year t, where $t = 10$ corresponds to 2010.

(a) Find the function $p(t)$ that represents the total number of dogs and cats in the United States.

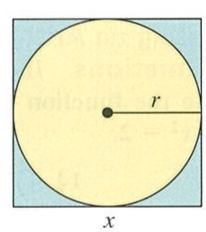
(b) Interpret $p(16)$.

(c) Let $n(t)$ represent the population of the United States in year t, where $t = 10$ corresponds to 2010. Find and interpret

$h(t) = p(t)/n(t)$.



61. Geometry A square concrete foundation is a base for a cylindrical tank (see figure).



(a) Write the radius r of the tank as a function of the length x of the sides of the square.

(b) Write the area A of the circular base of the tank as a function of the radius r.

(c) Find and interpret $(A \circ r)(x)$.

62. Biology The number N of bacteria in a refrigerated food is given by

$$N(T) = 10T^2 - 20T + 600, \quad 2 \le T \le 20$$

where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature of the food is given by

$$T(t) = 3t + 2, \quad 0 \le t \le 6$$

where t is the time in hours.

(a) Find and interpret $(N \circ T)(t)$.

(b) Find the bacteria count after 0.5 hour.

(c) Find the time when the bacteria count reaches 1500.

63. Salary You are a sales representative for a clothing manufacturer. You are paid an annual salary, plus a bonus of 3% of your sales over $500,000. Consider the two functions $f(x) = x - 500{,}000$ and $g(x) = 0.03x$. When x is greater than $500,000, which of the following represents your bonus? Explain.

(a) $f(g(x))$

(b) $g(f(x))$

64. Consumer Awareness The suggested retail price of a new hybrid car is p dollars. The dealership advertises a factory rebate of $2000 and a 10% discount.

(a) Write a function R in terms of p giving the cost of the hybrid car after receiving the rebate from the factory.

(b) Write a function S in terms of p giving the cost of the hybrid car after receiving the dealership discount.

(c) Find and interpret $(R \circ S)(p)$ and $(S \circ R)(p)$.

(d) Find $(R \circ S)(25{,}795)$ and $(S \circ R)(25{,}795)$. Which yields the lower cost for the hybrid car? Explain.

Exploration

True or False? In Exercises 65 and 66, determine whether the statement is true or false. Justify your answer.

65. If $f(x) = x + 1$ and $g(x) = 6x$, then

$$(f \circ g)(x) = (g \circ f)(x).$$

66. When you are given two functions f and g and a constant c, you can find $(f \circ g)(c)$ if and only if $g(c)$ is in the domain of f.

Siblings In Exercises 67 and 68, three siblings are three different ages. The oldest is twice the age of the middle sibling, and the middle sibling is six years older than one-half the age of the youngest.

67. (a) Write a composite function that gives the oldest sibling's age in terms of the youngest. Explain how you arrived at your answer.

(b) If the oldest sibling is 16 years old, find the ages of the other two siblings.

68. (a) Write a composite function that gives the youngest sibling's age in terms of the oldest. Explain how you arrived at your answer.

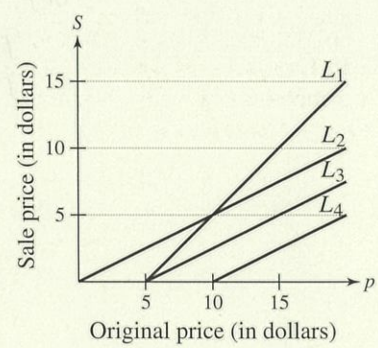
(b) If the youngest sibling is 2 years old, find the ages of the other two siblings.

69. Proof Prove that the product of two odd functions is an even function, and that the product of two even functions is an even function.

70. Conjecture Use examples to hypothesize whether the product of an odd function and an even function is even or odd. Then prove your hypothesis.

71. Writing Functions Write two unique functions f and g such that $(f \circ g)(x) = (g \circ f)(x)$ and f and g are (a) linear functions and (b) polynomial functions with degrees greater than one.

72. HOW DO YOU SEE IT? The graphs labeled L_1, L_2, L_3, and L_4 represent four different pricing discounts, where p is the original price (in dollars) and S is the sale price (in dollars). Match each function with its graph. Describe the situations in parts (c) and (d).



(a) $f(p)$: A 50% discount is applied.

(b) $g(p)$: A $5 discount is applied.

(c) $(g \circ f)(p)$

(d) $(f \circ g)(p)$

73. Proof

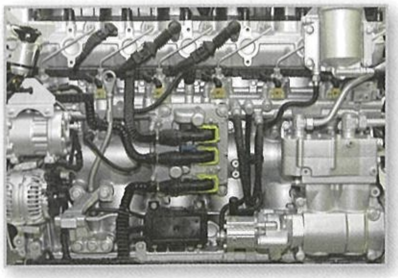
(a) Given a function f, prove that g is even and h is odd, where $g(x) = \frac{1}{2}[f(x) + f(-x)]$ and $h(x) = \frac{1}{2}[f(x) - f(-x)]$.

(b) Use the result of part (a) to prove that any function can be written as a sum of even and odd functions. [*Hint:* Add the two equations in part (a).]

(c) Use the result of part (b) to write each function as a sum of even and odd functions.

$$f(x) = x^2 - 2x + 1, \quad k(x) = \frac{1}{x + 1}$$

1.9 Inverse Functions



Inverse functions can help you model and solve real-life problems. For example, in Exercise 90 on page 92, you will write an inverse function and use it to determine the percent load interval for a diesel engine.

- **Find inverse functions informally and verify that two functions are inverse functions of each other.**
- **Use graphs to verify that two functions are inverse functions of each other.**
- **Use the Horizontal Line Test to determine whether functions are one-to-one.**
- **Find inverse functions algebraically.**

Inverse Functions

Recall from Section 1.4 that a set of ordered pairs can represent a function. For example, the function $f(x) = x + 4$ from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8\}$ can be written as

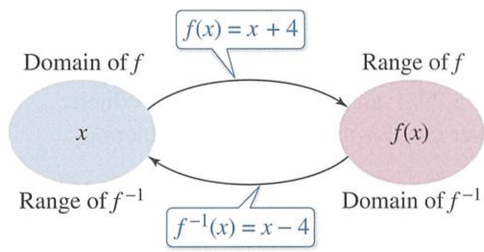
$$f(x) = x + 4: \{(1, 5), (2, 6), (3, 7), (4, 8)\}.$$

In this case, by interchanging the first and second coordinates of each of the ordered pairs, you form the **inverse function** of f, which is denoted by f^{-1}. It is a function from the set B to the set A, and can be written as

$$f^{-1}(x) = x - 4: \{(5, 1), (6, 2), (7, 3), (8, 4)\}.$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in the figure below. Also note that the functions f and f^{-1} have the effect of "undoing" each other. In other words, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = f(x - 4) = (x - 4) + 4 = x$$
$$f^{-1}(f(x)) = f^{-1}(x + 4) = (x + 4) - 4 = x$$



EXAMPLE 1 Finding an Inverse Function Informally

Find the inverse function of $f(x) = 4x$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution The function f *multiplies* each input by 4. To "undo" this function, you need to *divide* each input by 4. So, the inverse function of $f(x) = 4x$ is

$$f^{-1}(x) = \frac{x}{4}.$$

Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

$$f(f^{-1}(x)) = f\left(\frac{x}{4}\right) = 4\left(\frac{x}{4}\right) = x \qquad f^{-1}(f(x)) = f^{-1}(4x) = \frac{4x}{4} = x$$

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Find the inverse function of $f(x) = \frac{1}{5}x$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Definition of Inverse Function

Let f and g be two functions such that

$$f(g(x)) = x \qquad \text{for every } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \qquad \text{for every } x \text{ in the domain of } f.$$

Under these conditions, the function g is the **inverse function** of the function f. The function g is denoted by f^{-1} (read "f-inverse"). So,

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

Do not be confused by the use of -1 to denote the inverse function f^{-1}. In this text, whenever f^{-1} is written, it *always* refers to the inverse function of the function f and *not* to the reciprocal of $f(x)$.

If the function g is the inverse function of the function f, then it must also be true that the function f is the inverse function of the function g. So, it is correct to say that the functions f and g are *inverse functions of each other.*

EXAMPLE 2 Verifying Inverse Functions

Which of the functions is the inverse function of $f(x) = \dfrac{5}{x-2}$?

$$g(x) = \frac{x-2}{5} \qquad h(x) = \frac{5}{x} + 2$$

Solution By forming the composition of f with g, you have

$$f(g(x)) = f\left(\frac{x-2}{5}\right) = \frac{5}{\left(\frac{x-2}{5}\right) - 2} = \frac{25}{x-12} \neq x.$$

This composition is not equal to the identity function x, so g *is not* the inverse function of f. By forming the composition of f with h, you have

$$f(h(x)) = f\left(\frac{5}{x} + 2\right) = \frac{5}{\left(\frac{5}{x} + 2\right) - 2} = \frac{5}{\left(\frac{5}{x}\right)} = x.$$

So, it appears that h *is* the inverse function of f. Confirm this by showing that the composition of h with f is also equal to the identity function.

$$h(f(x)) = h\left(\frac{5}{x-2}\right) = \frac{5}{\left(\frac{5}{x-2}\right)} + 2 = x - 2 + 2 = x$$

Check to see that the domain of f is the same as the range of h and vice versa.

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Which of the functions is the inverse function of $f(x) = \dfrac{x-4}{7}$?

$$g(x) = 7x + 4 \qquad h(x) = \frac{7}{x-4}$$

The Graph of an Inverse Function

The graphs of a function f and its inverse function f^{-1} are related to each other in this way: If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa. This means that the graph of f^{-1} is a *reflection* of the graph of f in the line $y = x$, as shown in Figure 1.56.

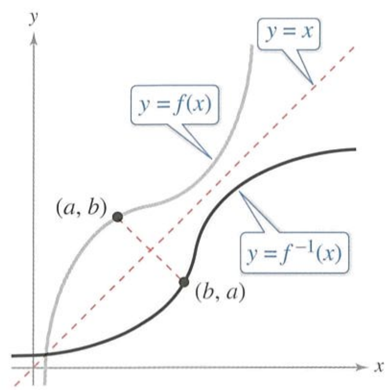

Figure 1.56

EXAMPLE 3 Verifying Inverse Functions Graphically

Verify graphically that the functions $f(x) = 2x - 3$ and $g(x) = \frac{1}{2}(x + 3)$ are inverse functions of each other.

Solution Sketch the graphs of f and g on the same rectangular coordinate system, as shown in Figure 1.57. It appears that the graphs are reflections of each other in the line $y = x$. Further verify this reflective property by testing a few points on each graph. Note that for each point (a, b) on the graph of f, the point (b, a) is on the graph of g.

Graph of $f(x) = 2x - 3$	Graph of $g(x) = \frac{1}{2}(x + 3)$
$(-1, -5)$	$(-5, -1)$
$(0, -3)$	$(-3, 0)$
$(1, -1)$	$(-1, 1)$
$(2, 1)$	$(1, 2)$
$(3, 3)$	$(3, 3)$

The graphs of f and g are reflections of each other in the line $y = x$. So, f and g are inverse functions of each other.

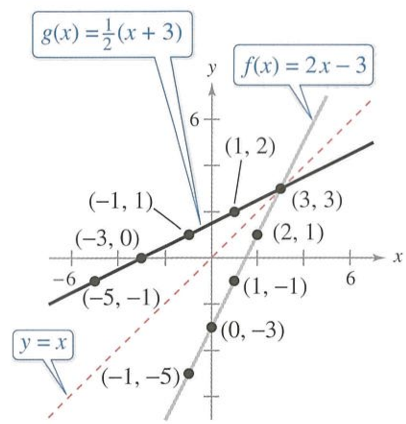

Figure 1.57

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Verify graphically that the functions $f(x) = 4x - 1$ and $g(x) = \frac{1}{4}(x + 1)$ are inverse functions of each other.

EXAMPLE 4 Verifying Inverse Functions Graphically

Verify graphically that the functions $f(x) = x^2$ $(x \geq 0)$ and $g(x) = \sqrt{x}$ are inverse functions of each other.

Solution Sketch the graphs of f and g on the same rectangular coordinate system, as shown in Figure 1.58. It appears that the graphs are reflections of each other in the line $y = x$. Test a few points on each graph.

Graph of $f(x) = x^2,\ x \geq 0$	Graph of $g(x) = \sqrt{x}$
$(0, 0)$	$(0, 0)$
$(1, 1)$	$(1, 1)$
$(2, 4)$	$(4, 2)$
$(3, 9)$	$(9, 3)$

The graphs of f and g are reflections of each other in the line $y = x$. So, f and g are inverse functions of each other.

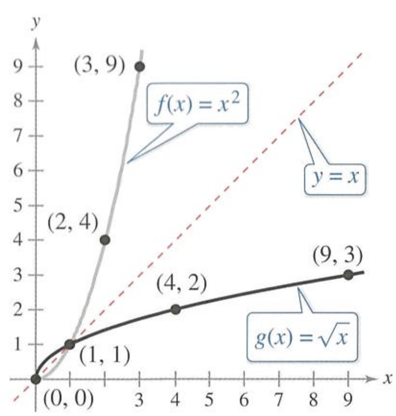

Figure 1.58

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Verify graphically that the functions $f(x) = x^2 + 1$ $(x \geq 0)$ and $g(x) = \sqrt{x - 1}$ are inverse functions of each other.

One-to-One Functions

The reflective property of the graphs of inverse functions gives you a graphical test for determining whether a function has an inverse function. This test is the **Horizontal Line Test** for inverse functions.

> **Horizontal Line Test for Inverse Functions**
>
> A function f has an inverse function if and only if no *horizontal* line intersects the graph of f at more than one point.

If no horizontal line intersects the graph of f at more than one point, then no y-value corresponds to more than one x-value. This is the essential characteristic of **one-to-one functions.**

> **One-to-One Functions**
>
> A function f is **one-to-one** when each value of the dependent variable corresponds to exactly one value of the independent variable. A function f has an inverse function if and only if f is one-to-one.

Consider the table of values for the function $f(x) = x^2$ on the left. The output $f(x) = 4$ corresponds to two inputs, $x = -2$ and $x = 2$, so f is not one-to-one. In the table on the right, x and y are interchanged. Here $x = 4$ corresponds to both $y = -2$ and $y = 2$, so this table does not represent a function. So, $f(x) = x^2$ is not one-to-one and does not have an inverse function.

x	$f(x) = x^2$
-2	4
-1	1
0	0
1	1
2	4
3	9

x	y
4	-2
1	-1
0	0
1	1
4	2
9	3

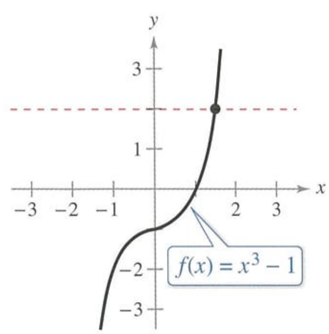

Figure 1.59

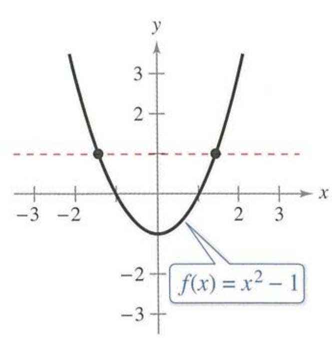

Figure 1.60

EXAMPLE 5 Applying the Horizontal Line Test

See LarsonPrecalculus.com for an interactive version of this type of example.

a. The graph of the function $f(x) = x^3 - 1$ is shown in Figure 1.59. No horizontal line intersects the graph of f at more than one point, so f *is* a one-to-one function and *does* have an inverse function.

b. The graph of the function $f(x) = x^2 - 1$ is shown in Figure 1.60. It is possible to find a horizontal line that intersects the graph of f at more than one point, so f *is not* a one-to-one function and *does not* have an inverse function.

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Use the graph of f to determine whether the function has an inverse function.

a. $f(x) = \frac{1}{2}(3 - x)$ **b.** $f(x) = |x|$

Finding Inverse Functions Algebraically

For relatively simple functions (such as the one in Example 1), you can find inverse functions by inspection. For more complicated functions, however, it is best to use the guidelines below. The key step in these guidelines is Step 3—interchanging the roles of x and y. This step corresponds to the fact that inverse functions have ordered pairs with the coordinates reversed.

REMARK Note what happens when you try to find the inverse function of a function that is not one-to-one.

$f(x) = x^2 + 1$ Original function

$y = x^2 + 1$ Replace $f(x)$ with y.

$x = y^2 + 1$ Interchange x and y.

$x - 1 = y^2$ Isolate y-term.

$y = \pm\sqrt{x - 1}$ Solve for y.

You obtain two y-values for each x.

Finding an Inverse Function

1. Use the Horizontal Line Test to decide whether f has an inverse function.
2. In the equation for $f(x)$, replace $f(x)$ with y.
3. Interchange the roles of x and y, and solve for y.
4. Replace y with $f^{-1}(x)$ in the new equation.
5. Verify that f and f^{-1} are inverse functions of each other by showing that the domain of f is equal to the range of f^{-1}, the range of f is equal to the domain of f^{-1}, and $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

EXAMPLE 6 Finding an Inverse Function Algebraically

Find the inverse function of

$$f(x) = \frac{5 - x}{3x + 2}.$$

Solution The graph of f is shown in Figure 1.61. This graph passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function.

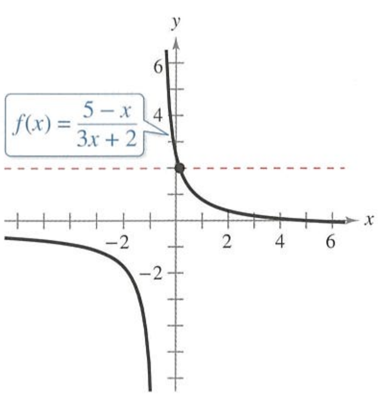


Figure 1.61

$$f(x) = \frac{5 - x}{3x + 2}$$ Write original function.

$$y = \frac{5 - x}{3x + 2}$$ Replace $f(x)$ with y.

$$x = \frac{5 - y}{3y + 2}$$ Interchange x and y.

$$x(3y + 2) = 5 - y$$ Multiply each side by $3y + 2$.

$$3xy + 2x = 5 - y$$ Distributive Property

$$3xy + y = 5 - 2x$$ Collect terms with y.

$$y(3x + 1) = 5 - 2x$$ Factor.

$$y = \frac{5 - 2x}{3x + 1}$$ Solve for y.

$$f^{-1}(x) = \frac{5 - 2x}{3x + 1}$$ Replace y with $f^{-1}(x)$.

Check that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Find the inverse function of

$$f(x) = \frac{5 - 3x}{x + 2}.$$

EXAMPLE 7 Finding an Inverse Function Algebraically

Find the inverse function of

$$f(x) = \sqrt{2x - 3}.$$

Solution The graph of f is shown in the figure below. This graph passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function.

$f(x) = \sqrt{2x - 3}$ Write original function.

$y = \sqrt{2x - 3}$ Replace $f(x)$ with y.

$x = \sqrt{2y - 3}$ Interchange x and y.

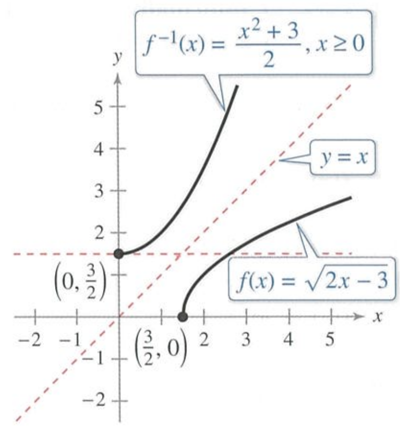
$x^2 = 2y - 3$ Square each side.

$2y = x^2 + 3$ Isolate y-term.

$y = \dfrac{x^2 + 3}{2}$ Solve for y.

$f^{-1}(x) = \dfrac{x^2 + 3}{2}, \ x \geq 0$ Replace y with $f^{-1}(x)$.

The graph of f^{-1} in the figure is the reflection of the graph of f in the line $y = x$. Note that the range of f is the interval $[0, \infty)$, which implies that the domain of f^{-1} is the interval $[0, \infty)$. Moreover, the domain of f is the interval $\left[\frac{3}{2}, \infty\right)$, which implies that the range of f^{-1} is the interval $\left[\frac{3}{2}, \infty\right)$. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.



Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Find the inverse function of

$$f(x) = \sqrt[3]{10 + x}.$$

Summarize (Section 1.9)

1. State the definition of an inverse function *(page 85)*. For examples of finding inverse functions informally and verifying inverse functions, see Examples 1 and 2.
2. Explain how to use graphs to verify that two functions are inverse functions of each other *(page 86)*. For examples of verifying inverse functions graphically, see Examples 3 and 4.
3. Explain how to use the Horizontal Line Test to determine whether a function is one-to-one *(page 87)*. For an example of applying the Horizontal Line Test, see Example 5.
4. Explain how to find an inverse function algebraically *(page 88)*. For examples of finding inverse functions algebraically, see Examples 6 and 7.

1.9 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. If $f(g(x))$ and $g(f(x))$ both equal x, then the function g is the ________ function of the function f.
2. The inverse function of f is denoted by ________.
3. The domain of f is the ________ of f^{-1}, and the ________ of f^{-1} is the range of f.
4. The graphs of f and f^{-1} are reflections of each other in the line ________.
5. A function f is ________ when each value of the dependent variable corresponds to exactly one value of the independent variable.
6. A graphical test for the existence of an inverse function of f is the _______ Line Test.

Skills and Applications

Finding an Inverse Function Informally

In Exercises 7–14, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

7. $f(x) = 6x$
8. $f(x) = \frac{1}{3}x$
9. $f(x) = 3x + 1$
10. $f(x) = \frac{x - 3}{2}$
11. $f(x) = x^2 - 4,\ x \geq 0$
12. $f(x) = x^2 + 2,\ x \geq 0$
13. $f(x) = x^3 + 1$
14. $f(x) = \frac{x^5}{4}$

Verifying Inverse Functions

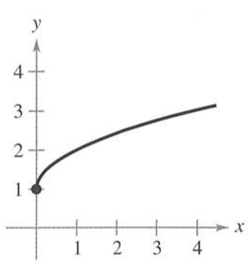
In Exercises 15–18, verify that f and g are inverse functions algebraically.

15. $f(x) = \frac{x - 9}{4},\quad g(x) = 4x + 9$
16. $f(x) = -\frac{3}{2}x - 4,\quad g(x) = -\frac{2x + 8}{3}$
17. $f(x) = \frac{x^3}{4},\quad g(x) = \sqrt[3]{4x}$
18. $f(x) = x^3 + 5,\quad g(x) = \sqrt[3]{x - 5}$

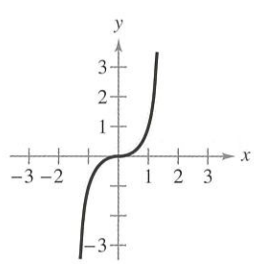
Sketching the Graph of an Inverse Function

In Exercises 19 and 20, use the graph of the function to sketch the graph of its inverse function $y = f^{-1}(x)$.

19.



20.



Verifying Inverse Functions

In Exercises 21–32, verify that f and g are inverse functions (a) algebraically and (b) graphically.

21. $f(x) = x - 5,\quad g(x) = x + 5$
22. $f(x) = 2x,\quad g(x) = \frac{x}{2}$
23. $f(x) = 7x + 1,\quad g(x) = \frac{x - 1}{7}$
24. $f(x) = 3 - 4x,\quad g(x) = \frac{3 - x}{4}$
25. $f(x) = x^3,\quad g(x) = \sqrt[3]{x}$
26. $f(x) = \frac{x^3}{3},\quad g(x) = \sqrt[3]{3x}$
27. $f(x) = \sqrt{x + 5},\quad g(x) = x^2 - 5,\quad x \geq 0$
28. $f(x) = 1 - x^3,\quad g(x) = \sqrt[3]{1 - x}$
29. $f(x) = \frac{1}{x},\quad g(x) = \frac{1}{x}$
30. $f(x) = \frac{1}{1 + x},\quad x \geq 0,\quad g(x) = \frac{1 - x}{x},\quad 0 < x \leq 1$
31. $f(x) = \frac{x - 1}{x + 5},\quad g(x) = -\frac{5x + 1}{x - 1}$
32. $f(x) = \frac{x + 3}{x - 2},\quad g(x) = \frac{2x + 3}{x - 1}$

Using a Table to Determine an Inverse Function

In Exercises 33 and 34, does the function have an inverse function?

33.

x	−1	0	1	2	3	4
$f(x)$	−2	1	2	1	−2	−6

34.

x	−3	−2	−1	0	2	3
$f(x)$	10	6	4	1	−3	−10

Using a Table to Find an Inverse Function In Exercises 35 and 36, use the table of values for $y = f(x)$ to complete a table for $y = f^{-1}(x)$.

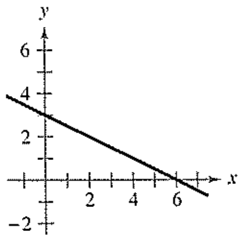
35.

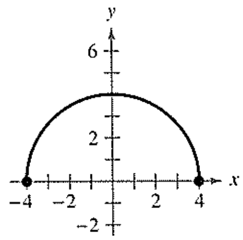
x	-1	0	1	2	3	4
$f(x)$	3	5	7	9	11	13

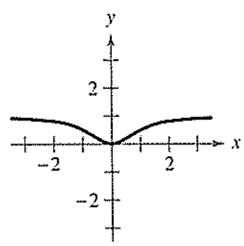
36.

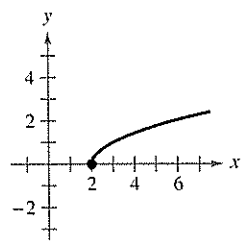
x	-3	-2	-1	0	1	2
$f(x)$	10	5	0	-5	-10	-15

Applying the Horizontal Line Test In Exercises 37–40, does the function have an inverse function?

37.


38.


39.


40.


Applying the Horizontal Line Test In Exercises 41–44, use a graphing utility to graph the function, and use the Horizontal Line Test to determine whether the function has an inverse function.

41. $g(x) = (x + 3)^2 + 2$
42. $f(x) = \frac{1}{5}(x + 2)^3$
43. $f(x) = x\sqrt{9 - x^2}$
44. $h(x) = |x| - |x - 4|$

Finding and Analyzing Inverse Functions In Exercises 45–54, (a) find the inverse function of f, (b) graph both f and f^{-1} on the same set of coordinate axes, (c) describe the relationship between the graphs of f and f^{-1}, and (d) state the domains and ranges of f and f^{-1}.

45. $f(x) = x^5 - 2$
46. $f(x) = x^3 + 8$
47. $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$
48. $f(x) = x^2 - 2, \quad x \le 0$
49. $f(x) = \dfrac{4}{x}$
50. $f(x) = -\dfrac{2}{x}$
51. $f(x) = \dfrac{x + 1}{x - 2}$
52. $f(x) = \dfrac{x - 2}{3x + 5}$
53. $f(x) = \sqrt[3]{x - 1}$
54. $f(x) = x^{3/5}$

Finding an Inverse Function In Exercises 55–70, determine whether the function has an inverse function. If it does, find the inverse function.

55. $f(x) = x^4$
56. $f(x) = \dfrac{1}{x^2}$
57. $g(x) = \dfrac{x + 1}{6}$
58. $f(x) = 3x + 5$
59. $p(x) = -4$
60. $f(x) = 0$
61. $f(x) = (x + 3)^2, \quad x \ge -3$
62. $q(x) = (x - 5)^2$
63. $f(x) = \begin{cases} x + 3, & x < 0 \\ 6 - x, & x \ge 0 \end{cases}$
64. $f(x) = \begin{cases} -x, & x \le 0 \\ x^2 - 3x, & x > 0 \end{cases}$
65. $h(x) = |x + 1| - 1$
66. $f(x) = |x - 2|, \quad x \le 2$
67. $f(x) = \sqrt{2x + 3}$
68. $f(x) = \sqrt{x - 2}$
69. $f(x) = \dfrac{6x + 4}{4x + 5}$
70. $f(x) = \dfrac{5x - 3}{2x + 5}$

Restricting the Domain In Exercises 71–78, restrict the domain of the function f so that the function is one-to-one and has an inverse function. Then find the inverse function f^{-1}. State the domains and ranges of f and f^{-1}. Explain your results. (There are many correct answers.)

71. $f(x) = |x + 2|$
72. $f(x) = |x - 5|$
73. $f(x) = (x + 6)^2$
74. $f(x) = (x - 4)^2$
75. $f(x) = -2x^2 + 5$
76. $f(x) = \frac{1}{2}x^2 - 1$
77. $f(x) = |x - 4| + 1$
78. $f(x) = -|x - 1| - 2$

Composition with Inverses In Exercises 79–84, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the value or function.

79. $(f^{-1} \circ g^{-1})(1)$
80. $(g^{-1} \circ f^{-1})(-3)$
81. $(f^{-1} \circ f^{-1})(4)$
82. $(g^{-1} \circ g^{-1})(-1)$
83. $(f \circ g)^{-1}$
84. $g^{-1} \circ f^{-1}$

Composition with Inverses In Exercises 85–88, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the function.

85. $g^{-1} \circ f^{-1}$
86. $f^{-1} \circ g^{-1}$
87. $(f \circ g)^{-1}$
88. $(g \circ f)^{-1}$

89. Hourly Wage Your wage is \$10.00 per hour plus \$0.75 for each unit produced per hour. So, your hourly wage y in terms of the number of units produced x is $y = 10 + 0.75x$.

(a) Find the inverse function. What does each variable represent in the inverse function?

(b) Determine the number of units produced when your hourly wage is \$24.25.

90. Diesel Mechanics The function

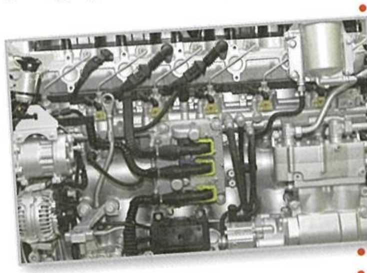
$$y = 0.03x^2 + 245.50, \quad 0 < x < 100$$

approximates the exhaust temperature y in degrees Fahrenheit, where x is the percent load for a diesel engine.

(a) Find the inverse function. What does each variable represent in the inverse function?

(b) Use a graphing utility to graph the inverse function.

(c) The exhaust temperature of the engine must not exceed 500 degrees Fahrenheit. What is the percent load interval?



Exploration

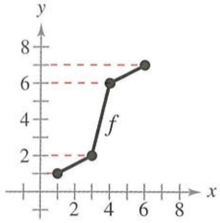
True or False? **In Exercises 91 and 92, determine whether the statement is true or false. Justify your answer.**

91. If f is an even function, then f^{-1} exists.

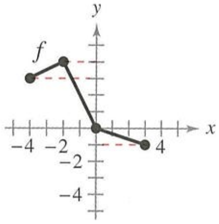
92. If the inverse function of f exists and the graph of f has a y-intercept, then the y-intercept of f is an x-intercept of f^{-1}.

Creating a Table **In Exercises 93 and 94, use the graph of the function f to create a table of values for the given points. Then create a second table that can be used to find f^{-1}, and sketch the graph of f^{-1}, if possible.**

93.



94.



95. Proof Prove that if f and g are one-to-one functions, then $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

96. Proof Prove that if f is a one-to-one odd function, then f^{-1} is an odd function.

97. Think About It The function $f(x) = k(2 - x - x^3)$ has an inverse function, and $f^{-1}(3) = -2$. Find k.

98. Think About It Consider the functions $f(x) = x + 2$ and $f^{-1}(x) = x - 2$. Evaluate $f(f^{-1}(x))$ and $f^{-1}(f(x))$ for the given values of x. What can you conclude about the functions?

x	-10	0	7	45
$f(f^{-1}(x))$				
$f^{-1}(f(x))$				

99. Think About It Restrict the domain of

$$f(x) = x^2 + 1$$

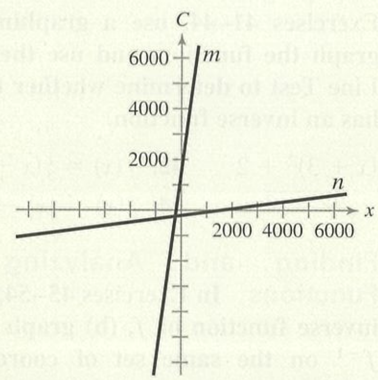
to $x \geq 0$. Use a graphing utility to graph the function. Does the restricted function have an inverse function? Explain.

100. HOW DO YOU SEE IT? The cost C for a business to make personalized T-shirts is given by

$$C(x) = 7.50x + 1500$$

where x represents the number of T-shirts.

(a) The graphs of C and C^{-1} are shown below. Match each function with its graph.



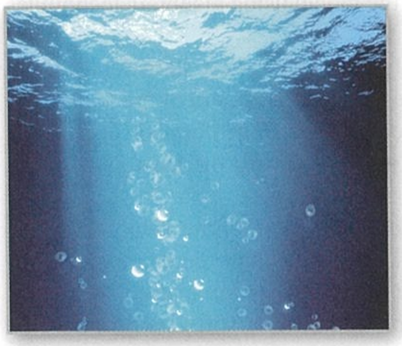
(b) Explain what $C(x)$ and $C^{-1}(x)$ represent in the context of the problem.

One-to-One Function Representation **In Exercises 101 and 102, determine whether the situation can be represented by a one-to-one function. If so, write a statement that best describes the inverse function.**

101. The number of miles n a marathon runner has completed in terms of the time t in hours

102. The depth of the tide d at a beach in terms of the time t over a 24-hour period

1.10 Mathematical Modeling and Variation



Mathematical models have a wide variety of real-life applications. For example, in Exercise 71 on page 103, you will use variation to model ocean temperatures at various depths.

- Use mathematical models to approximate sets of data points.
- Use the *regression* feature of a graphing utility to find equations of least squares regression lines.
- Write mathematical models for direct variation.
- Write mathematical models for direct variation as an *n*th power.
- Write mathematical models for inverse variation.
- Write mathematical models for combined variation.
- Write mathematical models for joint variation.

Introduction

In this section, you will study two techniques for fitting models to data: *least squares regression* and *direct and inverse variation.*

EXAMPLE 1 Using a Mathematical Model

The table shows the populations y (in millions) of the United States from 2008 through 2015. *(Source: U.S. Census Bureau)*

DATA

Year	2008	2009	2010	2011	2012	2013	2014	2015
Population, y	304.1	306.8	309.3	311.7	314.1	316.5	318.9	321.2

Spreadsheet at LarsonPrecalculus.com

A linear model that approximates the data is

$$y = 2.43t + 284.9, \quad 8 \le t \le 15$$

where t represents the year, with $t = 8$ corresponding to 2008. Plot the actual data *and* the model on the same graph. How closely does the model represent the data?

Solution Figure 1.62 shows the actual data and the model plotted on the same graph. From the graph, it appears that the model is a "good fit" for the actual data. To see how well the model fits, compare the actual values of y with the values of y found using the model. The values found using the model are labeled y^* in the table below.

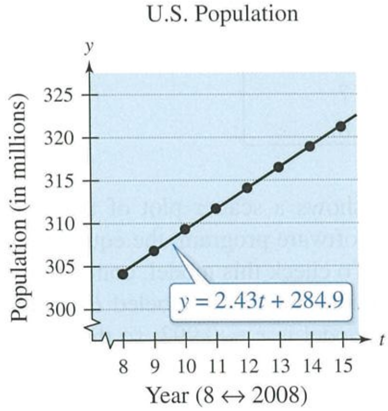


Figure 1.62

t	8	9	10	11	12	13	14	15
y	304.1	306.8	309.3	311.7	314.1	316.5	318.9	321.2
y^*	304.3	306.8	309.2	311.6	314.1	316.5	318.9	321.4

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The ordered pairs below give the median sales prices y (in thousands of dollars) of new homes sold in a neighborhood from 2009 through 2016. *(Spreadsheet at LarsonPrecalculus.com)*

DATA

(2009, 179.4)	(2011, 191.0)	(2013, 202.6)	(2015, 214.9)
(2010, 185.4)	(2012, 196.7)	(2014, 208.7)	(2016, 221.4)

A linear model that approximates the data is $y = 5.96t + 125.5$, $9 \le t \le 16$, where t represents the year, with $t = 9$ corresponding to 2009. Plot the actual data *and* the model on the same graph. How closely does the model represent the data?

Least Squares Regression and Graphing Utilities

So far in this text, you have worked with many different types of mathematical models that approximate real-life data. In some instances the model was given (as in Example 1), whereas in other instances you found the model using algebraic techniques or a graphing utility.

To find a model that approximates a set of data most accurately, statisticians use a measure called the **sum of the squared differences,** which is the sum of the squares of the differences between actual data values and model values. The "best-fitting" linear model, called the **least squares regression line,** is the one with the least sum of the squared differences.

Recall that you can approximate this line visually by plotting the data points and drawing the line that appears to best fit the data—or you can enter the data points into a graphing utility or software program and use the *linear regression* feature.

When you use the *regression* feature of a graphing utility or software program, an "r-value" may be output. This is the **correlation coefficient** of the data and gives a measure of how well the model fits the data. The closer the value of $|r|$ is to 1, the better the fit.

EXAMPLE 2 Finding a Least Squares Regression Line

See LarsonPrecalculus.com for an interactive version of this type of example.

The table shows the numbers E (in millions) of Medicare private health plan enrollees from 2008 through 2015. Construct a scatter plot that represents the data and find the equation of the least squares regression line for the data. *(Source: U.S. Centers for Medicare and Medicaid Services)*

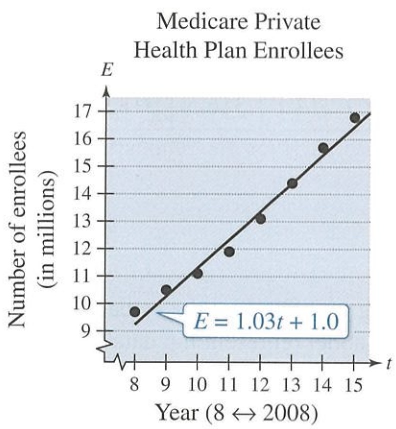


Figure 1.63

DATA

Spreadsheet at LarsonPrecalculus.com

Year	Enrollees, E
2008	9.7
2009	10.5
2010	11.1
2011	11.9
2012	13.1
2013	14.4
2014	15.7
2015	16.8

Solution Let $t = 8$ represent 2008. Figure 1.63 shows a scatter plot of the data. Using the *regression* feature of a graphing utility or software program, the equation of the least squares regression line is $E = 1.03t + 1.0$. To check this model, compare the actual E-values with the E-values found using the model, which are labeled E^* in the table at the left. The correlation coefficient for this model is $r \approx 0.992$, so the model is a good fit.

t	E	E^*
8	9.7	9.2
9	10.5	10.3
10	11.1	11.3
11	11.9	12.3
12	13.1	13.4
13	14.4	14.4
14	15.7	15.4
15	16.8	16.5

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

The ordered pairs below give the numbers E (in millions) of Medicare Advantage enrollees in health maintenance organization plans from 2008 through 2015. *(Spreadsheet at LarsonPrecalculus.com)* Construct a scatter plot that represents the data and find the equation of the least squares regression line for the data. *(Source: U.S. Centers for Medicare and Medicaid Services)*

DATA

(2008, 6.3) (2010, 7.2) (2012, 8.5) (2014, 10.1)
(2009, 6.7) (2011, 7.7) (2013, 9.3) (2015, 10.7)

Direct Variation

There are two basic types of linear models. The more general model has a nonzero y-intercept.

$$y = mx + b, \quad b \neq 0$$

The simpler model

$$y = kx$$

has a y-intercept of zero. In the simpler model, y **varies directly** as x, or is **directly proportional** to x.

> **Direct Variation**
>
> The statements below are equivalent.
>
> **1.** y **varies directly** as x.
>
> **2.** y is **directly proportional** to x.
>
> **3.** $y = kx$ for some nonzero constant k.
>
> k is the **constant of variation** or the **constant of proportionality.**

EXAMPLE 3 Direct Variation

In Pennsylvania, the state income tax is directly proportional to *gross income*. You work in Pennsylvania and your state income tax deduction is \$46.05 for a gross monthly income of \$1500. Find a mathematical model that gives the Pennsylvania state income tax in terms of gross income.

Solution

Verbal model: State income tax = k • Gross income

Labels: State income tax = y (dollars)
Gross income = x (dollars)
Income tax rate = k (percent in decimal form)

Equation: $y = kx$

To find the state income tax rate k, substitute the given information into the equation $y = kx$ and solve.

$y = kx$ — Write direct variation model.

$46.05 = k(1500)$ — Substitute 46.05 for y and 1500 for x.

$0.0307 = k$ — Divide each side by 1500.

So, the equation (or model) for state income tax in Pennsylvania is

$$y = 0.0307x.$$

In other words, Pennsylvania has a state income tax rate of 3.07% of gross income. Figure 1.64 shows the graph of this equation.

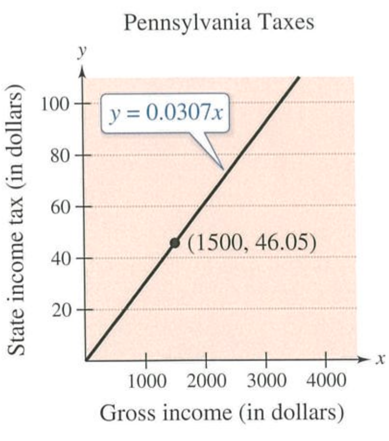


Figure 1.64

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

The simple interest on an investment is directly proportional to the amount of the investment. For example, an investment of \$2500 earns \$187.50 after 1 year. Find a mathematical model that gives the interest I after 1 year in terms of the amount invested P.

Direct Variation as an nth Power

Another type of direct variation relates one variable to a *power* of another variable. For example, in the formula for the area of a circle

$$A = \pi r^2$$

the area A is directly proportional to the square of the radius r. Note that for this formula, π is the constant of proportionality.

REMARK Note that the direct variation model $y = kx$ is a special case of $y = kx^n$ with $n = 1$.

Direct Variation as an nth Power

The statements below are equivalent.

1. y **varies directly as the nth power** of x.
2. y is **directly proportional to the nth power** of x.
3. $y = kx^n$ for some nonzero constant k.

EXAMPLE 4 Direct Variation as an nth Power

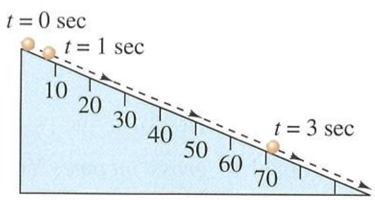


Figure 1.65

The distance a ball rolls down an inclined plane is directly proportional to the square of the time it rolls. During the first second, the ball rolls 8 feet. (See Figure 1.65.)

a. Write an equation relating the distance traveled to the time.

b. How far does the ball roll during the first 3 seconds?

Solution

a. Letting d be the distance (in feet) the ball rolls and letting t be the time (in seconds), you have

$$d = kt^2.$$

Now, $d = 8$ when $t = 1$, so you have

$d = kt^2$ Write direct variation model.

$8 = k(\ )^2$ Substitute 8 for d and 1 for t.

$8 = k$ Simplify.

and, the equation relating distance to time is

$$d = 8t^2.$$

b. When $t = 3$, the distance traveled is

$d = 8(3)^2$ Substitute 3 for t.

$= 8(9)$ Simplify.

$= 72$ feet. Simplify.

So, the ball rolls 72 feet during the first 3 seconds.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Neglecting air resistance, the distance s an object falls varies directly as the square of the duration t of the fall. An object falls a distance of 144 feet in 3 seconds. How far does it fall in 6 seconds?

In Examples 3 and 4, the direct variations are such that an *increase* in one variable corresponds to an *increase* in the other variable. You should not, however, assume that this always occurs with direct variation. For example, for the model $y = -3x$, an increase in x results in a *decrease* in y, and yet y is said to vary directly as x.

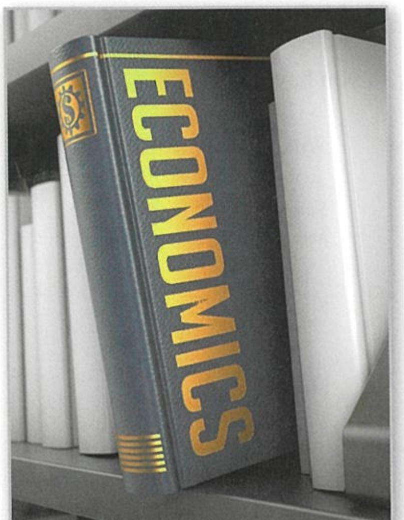
Inverse Variation

Inverse Variation

The statements below are equivalent.

1. y **varies inversely** as x.
2. y is **inversely proportional** to x.
3. $y = \frac{k}{x}$ for some nonzero constant k.

If x and y are related by an equation of the form $y = k/x^n$, then y varies inversely as the nth power of x (or y is inversely proportional to the nth power of x).



Supply and demand are fundamental concepts in economics. The law of demand states that, all other factors remaining equal, the lower the price of the product, the higher the quantity demanded. The law of supply states that the higher the price of the product, the higher the quantity supplied. *Equilibrium* occurs when the demand and the supply are the same.

EXAMPLE 5 Inverse Variation

A company has found that the demand for one of its products varies inversely as the price of the product. When the price is \$6.25, the demand is 400 units. Approximate the demand when the price is \$5.75.

Solution

Let p be the price and let x be the demand. The demand varies inversely as the price, so you have

$$x = \frac{k}{p}.$$

Now, $x = 400$ when $p = 6.25$, so you have

$$x = \frac{k}{p}$$ Write inverse variation model.

$$400 = \frac{k}{6.25}$$ Substitute 400 for x and 6.25 for p.

$$(400)(6.25) = k$$ Multiply each side by 6.25.

$$2500 = k$$ Simplify.

and the equation relating price and demand is

$$x = \frac{2500}{p}.$$

When $p = 5.75$, the demand is

$$x = \frac{2500}{p}$$ Write inverse variation model.

$$= \frac{2500}{5.75}$$ Substitute 5.75 for p.

$$\approx 435 \text{ units.}$$ Simplify.

So, the demand for the product is about 435 units when the price is \$5.75.

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

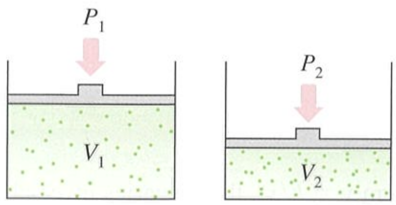
The company in Example 5 has found that the demand for another of its products also varies inversely as the price of the product. When the price is \$2.75, the demand is 600 units. Approximate the demand when the price is \$3.25.

Tashatuvango/Shutterstock.com

Combined Variation

Some applications of variation involve problems with *both* direct and inverse variations in the same model. These types of models have **combined variation.**

EXAMPLE 6 Combined Variation



If $P_2 > P_1$, then $V_2 < V_1$.

If the temperature is held constant and pressure increases, then the volume *decreases.*

Figure 1.66

A gas law states that the volume of an enclosed gas varies inversely as the pressure (Figure 1.66) *and* directly as the temperature. The pressure of a gas is 0.75 kilogram per square centimeter when the temperature is 294 K and the volume is 8000 cubic centimeters.

a. Write an equation relating pressure, temperature, and volume.

b. Find the pressure when the temperature is 300 K and the volume is 7000 cubic centimeters.

Solution

a. Volume V varies directly as temperature T and inversely as pressure P, so you have

$$V = \frac{kT}{P}.$$

Now, $P = 0.75$ when $T = 294$ and $V = 8000$, so you have

$$V = \frac{kT}{P}$$ Write combined variation model.

$$8000 = \frac{k(294)}{0.75}$$ Substitute 8000 for V, 294 for T, and 0.75 for P.

$$\frac{6000}{294} = k$$ Simplify.

$$\frac{1000}{49} = k$$ Simplify.

and the equation relating pressure, temperature, and volume is

$$V = \frac{1000}{49}\left(\frac{T}{P}\right).$$

b. Isolate P on one side of the equation by multiplying each side by P and dividing each side by V to obtain $P = \frac{1000}{49}\left(\frac{T}{V}\right)$. When $T = 300$ and $V = 7000$, the pressure is

$$P = \frac{1000}{49}\left(\frac{T}{V}\right)$$ Combined variation model solved for P.

$$= \frac{1000}{49}\left(\frac{300}{7000}\right)$$ Substitute 300 for T and 7000 for V.

$$= \frac{300}{343}$$ Simplify.

≈ 0.87 kilogram per square centimeter. Simplify.

So, the pressure is about 0.87 kilogram per square centimeter when the temperature is 300 K and the volume is 7000 cubic centimeters.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

The resistance of a copper wire carrying an electrical current is directly proportional to its length and inversely proportional to its cross-sectional area. A copper wire with a diameter of 0.0126 inch has a resistance of 64.9 ohms per thousand feet. What length of 0.0201-inch-diameter copper wire will produce a resistance of 33.5 ohms?

Joint Variation

> **Joint Variation**
>
> The statements below are equivalent.
>
> 1. z **varies jointly** as x and y.
> 2. z is **jointly proportional** to x and y.
> 3. $z = kxy$ for some nonzero constant k.

If x, y, and z are related by an equation of the form $z = kx^n y^m$, then z varies jointly as the nth power of x and the mth power of y.

EXAMPLE 7 Joint Variation

The *simple* interest for an investment is jointly proportional to the time and the principal. After one quarter (3 months), the interest on a principal of \$5000 is \$43.75. (a) Write an equation relating the interest, principal, and time. (b) Find the interest after three quarters.

Solution

a. Interest I (in dollars) is jointly proportional to principal P (in dollars) and time t (in years), so you have

$$I = kPt.$$

For $I = 43.75$, $P = 5000$, and $t = \frac{3}{12} = \frac{1}{4}$, you have $43.75 = k(5000)\left(\frac{1}{4}\right)$, which implies that $k = 4(43.75)/5000 = 0.035$. So, the equation relating interest, principal, and time is

$$I = 0.035Pt$$

which is the familiar equation for simple interest where the constant of proportionality, 0.035, represents an annual interest rate of 3.5%.

b. When $P = \$5000$ and $t = \frac{3}{4}$, the interest is $I = (0.035)(5000)\left(\frac{3}{4}\right) = \$131.25$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

The kinetic energy E of an object varies jointly with the object's mass m and the square of the object's velocity v. An object with a mass of 50 kilograms traveling at 16 meters per second has a kinetic energy of 6400 joules. What is the kinetic energy of an object with a mass of 70 kilograms traveling at 20 meters per second?

> **Summarize** (Section 1.10)
>
> 1. Explain how to use a mathematical model to approximate a set of data points *(page 93)*. For an example of using a mathematical model to approximate a set of data points, see Example 1.
> 2. Explain how to use the *regression* feature of a graphing utility to find the equation of a least squares regression line *(page 94)*. For an example of finding the equation of a least squares regression line, see Example 2.
> 3. Explain how to write mathematical models for direct variation, direct variation as an nth power, inverse variation, combined variation, and joint variation *(pages 95–99)*. For examples of these types of variation, see Examples 3–7.

1.10 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. Two techniques for fitting models to data are direct and inverse ________ and least squares ________.
2. Statisticians use a measure called the ________ of the ________ ________ to find a model that approximates a set of data most accurately.
3. The linear model with the least sum of the squared differences is called the ________ ________ ________ line.
4. An r-value, or ________ ________, of a set of data gives a measure of how well a model fits the data.
5. The direct variation model $y = kx^n$ can be described as "y varies directly as the nth power of x," or "y is ________ ________ to the nth power of x."
6. The mathematical model $y = \frac{2}{x}$ is an example of ________ variation.
7. Mathematical models that involve both direct and inverse variation have ________ variation.
8. The joint variation model $z = kxy$ can be described as "z varies jointly as x and y," or "z is ________ ________ to x and y."

Skills and Applications

Mathematical Models **In Exercises 9 and 10, (a) plot the actual data and the model of the same graph and (b) describe how closely the model represents the data. If the model does not closely represent the data, suggest another type of model that may be a better fit.**

9. The ordered pairs below give the civilian noninstitutional U.S. populations y (in millions of people) 16 years of age and over not in the civilian labor force from 2006 through 2014. *(Spreadsheet at LarsonPrecalculus.com)*

DATA

(2006, 77.4)	(2011, 86.0)
(2007, 78.7)	(2012, 88.3)
(2008, 79.5)	(2013, 90.3)
(2009, 81.7)	(2014, 92.0)
(2010, 83.9)	

A model for the data is $y = 1.92t + 65.0$, $6 \le t \le 14$, where t represents the years, with $t = 6$ corresponding to 2006. *(Source: U.S. Bureau of Labor Statistics)*

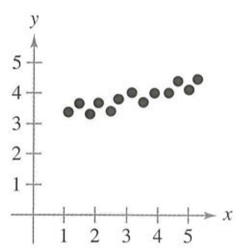
10. The ordered pairs below give the revenues y (in billions of dollars) for Activision Blizzard, Inc., from 2008 through 2014. *(Spreadsheet at LarsonPrecalculus.com)*

DATA

(2008, 3.03)	(2012, 4.86)
(2009, 4.28)	(2013, 4.58)
(2010, 4.45)	(2014, 4.41)
(2011, 4.76)	

A model for the data is $y = 0.184t + 2.32$, $8 \le t \le 14$, where t represents the year, with $t = 8$ corresponding to 2008. *(Source: Activision Blizzard, Inc.)*

Sketching a Line **In Exercises 11–16, sketch the line that you think best approximates the data in the scatter plot. Then find an equation of the line. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

11.


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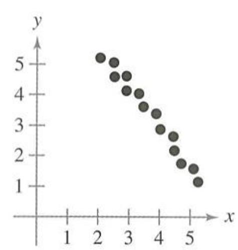

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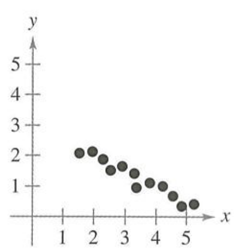
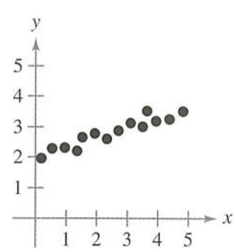
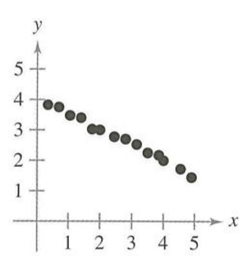
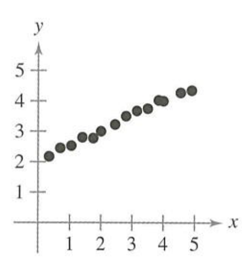


14.


15.


16.


17. Sports The ordered pairs below give the winning times (in seconds) of the women's 100-meter freestyle in the Olympics from 1984 through 2012. *(Spreadsheet at LarsonPrecalculus.com)* *(Source: International Olympic Committee)*

DATA

(1984, 55.92)	(2000, 53.83)
(1988, 54.93)	(2004, 53.84)
(1992, 54.64)	(2008, 53.12)
(1996, 54.50)	(2012, 53.00)

(a) Sketch a scatter plot of the data. Let y represent the winning time (in seconds) and let $t = 84$ represent 1984.

(b) Sketch the line that you think best approximates the data and find an equation of the line.

(c) Use the *regression* feature of a graphing utility to find the equation of the least squares regression line that fits the data.

(d) Compare the linear model you found in part (b) with the linear model you found in part (c).

18. Broadway The ordered pairs below give the starting year and gross ticket sales S (in millions of dollars) for each Broadway season in New York City from 1997 through 2014. *(Spreadsheet at LarsonPrecalculus.com)* *(Source: The Broadway League)*

DATA

(1997, 558)	(2003, 771)	(2009, 1020)
(1998, 588)	(2004, 769)	(2010, 1081)
(1999, 603)	(2005, 862)	(2011, 1139)
(2000, 666)	(2006, 939)	(2012, 1139)
(2001, 643)	(2007, 938)	(2013, 1269)
(2002, 721)	(2008, 943)	(2014, 1365)

(a) Use a graphing utility to create a scatter plot of the data. Let $t = 7$ represent 1997.

(b) Use the *regression* feature of the graphing utility to find the equation of the least squares regression line that fits the data.

(c) Use the graphing utility to graph the scatter plot you created in part (a) and the model you found in part (b) in the same viewing window. How closely does the model represent the data?

(d) Use the model to predict the gross ticket sales during the season starting in 2021.

(e) Interpret the meaning of the slope of the linear model in the context of the problem.

Direct Variation **In Exercises 19–24, find a direct variation model that relates y and x.**

19. $x = 2, y = 14$ **20.** $x = 5, y = 12$

21. $x = 5, y = 1$ **22.** $x = -24, y = 3$

23. $x = 4, y = 8\pi$ **24.** $x = \pi, y = -1$

Direct Variation as an *n*th Power **In Exercises 25–28, use the given values of k and n to complete the table for the direct variation model $y = kx^n$. Plot the points in a rectangular coordinate system.**

x	2	4	6	8	10
$y = kx^n$					

25. $k = 1, n = 2$ **26.** $k = 2, n = 2$

27. $k = \frac{1}{2}, n = 3$ **28.** $k = \frac{1}{4}, n = 3$

Inverse Variation as an *n*th Power **In Exercises 29–32, use the given values of k and n to complete the table for the inverse variation model $y = k/x^n$. Plot the points in a rectangular coordinate system.**

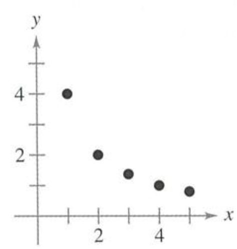
x	2	4	6	8	10
$y = k/x^n$					

29. $k = 2, n = 1$ **30.** $k = 5, n = 1$

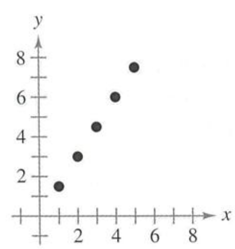
31. $k = 10, n = 2$ **32.** $k = 20, n = 2$

Think About It **In Exercises 33 and 34, use the graph to determine whether y varies directly as some power of x or inversely as some power of x. Explain.**

33.



34.



Determining Variation **In Exercises 35–38, determine whether the variation model represented by the ordered pairs (x, y) is of the form $y = kx$ or $y = k/x$, and find k. Then write a model that relates y and x.**

35. $(5, 1), \left(10, \frac{1}{2}\right), \left(15, \frac{1}{3}\right), \left(20, \frac{1}{4}\right), \left(25, \frac{1}{5}\right)$

36. $(5, 2), (10, 4), (15, 6), (20, 8), (25, 10)$

37. $(5, -3.5), (10, -7), (15, -10.5), (20, -14), (25, -17.5)$

38. $(5, 24), (10, 12), (15, 8), (20, 6), \left(25, \frac{24}{5}\right)$

Finding a Mathematical Model In Exercises 39–48, find a mathematical model for the verbal statement.

39. A varies directly as the square of r.
40. V varies directly as the cube of l.
41. y varies inversely as the square of x.
42. h varies inversely as the square root of s.
43. F varies directly as g and inversely as r^2.
44. z varies jointly as the square of x and the cube of y.
45. *Newton's Law of Cooling:* The rate of change R of the temperature of an object is directly proportional to the difference between the temperature T of the object and the temperature T_e of the environment.
46. *Boyle's Law:* For a constant temperature, the pressure P of a gas is inversely proportional to the volume V of the gas.
47. *Direct Current:* The electric power P of a direct current circuit is jointly proportional to the voltage V and the electric current I.
48. *Newton's Law of Universal Gravitation:* The gravitational attraction F between two objects of masses m_1 and m_2 is jointly proportional to the masses and inversely proportional to the square of the distance r between the objects.

Describing a Formula In Exercises 49–52, use variation terminology to describe the formula.

49. $y = 2x^2$
50. $t = \dfrac{72}{r}$
51. $A = \frac{1}{2}bh$
52. $K = \frac{1}{2}mv^2$

Finding a Mathematical Model In Exercises 53–60, find a mathematical model that represents the statement. (Determine the constant of proportionality.)

53. y is directly proportional to x. ($y = 54$ when $x = 3$.)
54. A varies directly as r^2. ($A = 9\pi$ when $r = 3$.)
55. y varies inversely as x. ($y = 3$ when $x = 25$.)
56. y is inversely proportional to x^3. ($y = 7$ when $x = 2$.)
57. z varies jointly as x and y. ($z = 64$ when $x = 4$ and $y = 8$.)
58. F is jointly proportional to r and the third power of s. ($F = 4158$ when $r = 11$ and $s = 3$.)
59. P varies directly as x and inversely as the square of y. $\left(P = \frac{28}{3}\right.$ when $x = 42$ and $\left.y = 9.\right)$
60. z varies directly as the square of x and inversely as y. ($z = 6$ when $x = 6$ and $y = 4$.)
61. **Simple Interest** The simple interest on an investment is directly proportional to the amount of the investment. An investment of \$3250 earns \$113.75 after 1 year. Find a mathematical model that gives the interest I after 1 year in terms of the amount invested P.
62. **Simple Interest** The simple interest on an investment is directly proportional to the amount of the investment. An investment of \$6500 earns \$211.25 after 1 year. Find a mathematical model that gives the interest I after 1 year in terms of the amount invested P.
63. **Measurement** Use the fact that 13 inches is approximately the same length as 33 centimeters to find a mathematical model that relates centimeters y to inches x. Then use the model to find the numbers of centimeters in 10 inches and 20 inches.
64. **Measurement** Use the fact that 14 gallons is approximately the same amount as 53 liters to find a mathematical model that relates liters y to gallons x. Then use the model to find the numbers of liters in 5 gallons and 25 gallons.

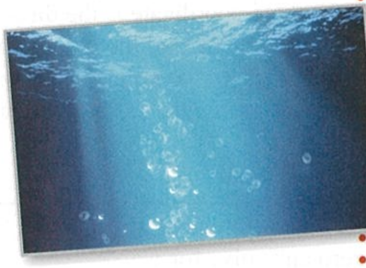
Hooke's Law In Exercises 65–68, use Hooke's Law, which states that the distance a spring stretches (or compresses) from its natural, or equilibrium, length varies directly as the applied force on the spring.

65. A force of 220 newtons stretches a spring 0.12 meter. What force stretches the spring 0.16 meter?
66. A force of 265 newtons stretches a spring 0.15 meter.
    (a) What force stretches the spring 0.1 meter?
    (b) How far does a force of 90 newtons stretch the spring?
67. The coiled spring of a toy supports the weight of a child. The weight of a 25-pound child compresses the spring a distance of 1.9 inches. The toy does not work properly when a weight compresses the spring more than 3 inches. What is the maximum weight for which the toy works properly?
68. An overhead garage door has two springs, one on each side of the door. A force of 15 pounds is required to stretch each spring 1 foot. Because of a pulley system, the springs stretch only one-half the distance the door travels. The door moves a total of 8 feet, and the springs are at their natural lengths when the door is open. Find the combined lifting force applied to the door by the springs when the door is closed.
69. **Ecology** The diameter of the largest particle that a stream can move is approximately directly proportional to the square of the velocity of the stream. When the velocity is $\frac{1}{4}$ mile per hour, the stream can move coarse sand particles about 0.02 inch in diameter. Approximate the velocity required to carry particles 0.12 inch in diameter.

70. Work The work W required to lift an object varies jointly with the object's mass m and the height h that the object is lifted. The work required to lift a 120-kilogram object 1.8 meters is 2116.8 joules. Find the amount of work required to lift a 100-kilogram object 1.5 meters.

71. Ocean Temperatures

The ordered pairs below give the average water temperatures C (in degrees Celsius) at several depths d (in meters) in the Indian Ocean. *(Spreadsheet at LarsonPrecalculus.com)* *(Source: NOAA)*



DATA

(1000, 4.85)	(2500, 1.888)
(1500, 3.525)	(3000, 1.583)
(2000, 2.468)	(3500, 1.422)

(a) Sketch a scatter plot of the data.

(b) Determine whether a direct variation model or an inverse variation model better fits the data.

(c) Find k for each pair of coordinates. Then find the mean value of k to find the constant of proportionality for the model you chose in part (b).

(d) Use your model to approximate the depth at which the water temperature is 3°C.

72. Light Intensity The ordered pairs below give the intensities y (in microwatts per square centimeter) of the light measured by a light probe located x centimeters from a light source. *(Spreadsheet at LarsonPrecalculus.com)*

DATA

(30, 0.1881)	(38, 0.1172)	(46, 0.0775)
(34, 0.1543)	(42, 0.0998)	(50, 0.0645)

A model that approximates the data is $y = 171.33/x^2$.

(a) Use a graphing utility to plot the data points and the model in the same viewing window.

(b) Use the model to approximate the light intensity 25 centimeters from the light source.

73. Music The fundamental frequency (in hertz) of a piano string is directly proportional to the square root of its tension and inversely proportional to its length and the square root of its mass density. A string has a frequency of 100 hertz. Find the frequency of a string with each property.

(a) Four times the tension

(b) Twice the length

(c) Four times the tension and twice the length

74. Beam Load The maximum load that a horizontal beam can safely support varies jointly as the width of the beam and the square of its depth and inversely as the length of the beam. Determine how each change affects the beam's maximum load.

(a) Doubling the width

(b) Doubling the depth

(c) Halving the length

(d) Halving the width and doubling the length

Exploration

True or False? **In Exercises 75 and 76, decide whether the statement is true or false. Justify your answer.**

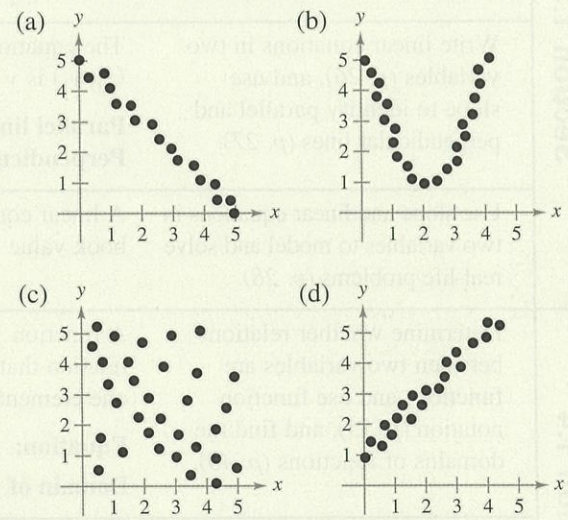
75. If y is directly proportional to x and x is directly proportional to z, then y is directly proportional to z.

76. If y is inversely proportional to x and x is inversely proportional to z, then y is inversely proportional to z.

77. Error Analysis Describe the error.

In the equation for the surface area of a sphere, $S = 4\pi r^2$, the surface area S varies jointly with π and the square of the radius r.

78. HOW DO YOU SEE IT? Discuss how well a linear model approximates the data shown in each scatter plot.



79. Think About It Let $y = 2x + 2$ and $t = x + 1$. What kind of variation do y and t have? Explain.

Project: Fraud and Identity Theft To work an extended application analyzing the numbers of fraud complaints and identity theft victims in the United States in 2014, visit this text's website at *LarsonPrecalculus.com.* *(Source: U.S. Federal Trade Commission)*

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Chapter Summary

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 1.1	Plot points in the Cartesian plane *(p. 2)*, use the Distance Formula *(p. 4)* and the Midpoint Formula *(p. 5)*, and use a coordinate plane to model and solve real-life problems *(p. 6)*.	For an ordered pair (x, y), the x-coordinate is the directed distance from the y-axis to the point, and the y-coordinate is the directed distance from the x-axis to the point. The coordinate plane can be used to estimate the annual sales of a company. (See Example 7.)	1–6
Section 1.2	Sketch graphs of equations *(p. 11)*, find x- and y-intercepts *(p. 14)*, and use symmetry to sketch graphs of equations *(p. 15)*.	To find x-intercepts, let y be zero and solve for x. To find y-intercepts, let x be zero and solve for y. Graphs can have symmetry with respect to one of the coordinate axes or with respect to the origin.	7–22
	Write equations of circles *(p. 17)*.	A point (x, y) lies on the circle of radius r and center (h, k) if and only if $(x - h)^2 + (y - k)^2 = r^2$.	23–27
	Use graphs of equations to solve real-life problems *(p. 18)*.	The graph of an equation can be used to estimate the maximum weight for a man in the U.S. Marine Corps. (See Example 9.)	28
Section 1.3	Use slope to graph linear equations in two variables *(p. 22)*.	The graph of the equation $y = mx + b$ is a line whose slope is m and whose y-intercept is $(0, b)$.	29–32
	Find the slope of a line given two points on the line *(p. 24)*.	The slope m of the nonvertical line through (x_1, y_1) and (x_2, y_2) is $m = (y_2 - y_1)/(x_2 - x_1)$, where $x_1 \neq x_2$.	33, 34
	Write linear equations in two variables *(p. 26)*, and use slope to identify parallel and perpendicular lines *(p. 27)*.	The equation of the line with slope m passing through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$. **Parallel lines:** $m_1 = m_2$ **Perpendicular lines:** $m_1 = -1/m_2$	35–40
	Use slope and linear equations in two variables to model and solve real-life problems *(p. 28)*.	A linear equation in two variables can help you describe the book value of exercise equipment each year. (See Example 7.)	41, 42
Section 1.4	Determine whether relations between two variables are functions and use function notation *(p. 35)*, and find the domains of functions *(p. 40)*.	A function f from a set A (domain) to a set B (range) is a relation that assigns to each element x in the set A exactly one element y in the set B. **Equation:** $f(x) = 5 - x^2$ **$f(2)$:** $f(2) = 5 - 2^2 = 1$ **Domain of $f(x) = 5 - x^2$:** All real numbers	43–50
	Use functions to model and solve real-life problems *(p. 41)*.	A function can model the path of a baseball. (See Example 9.)	51, 52
	Evaluate difference quotients *(p. 42)*.	**Difference quotient:** $\dfrac{f(x + h) - f(x)}{h}, \quad h \neq 0$	53, 54
Section 1.5	Use the Vertical Line Test for functions *(p. 50)*.	A set of points in a coordinate plane is the graph of y as a function of x if and only if no *vertical* line intersects the graph at more than one point.	55, 56
	Find the zeros of functions *(p. 51)*.	**Zeros of $y = f(x)$:** x-values for which $f(x) = 0$	57, 58

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 1.5	Determine intervals on which functions are increasing or decreasing *(p. 52)*, relative minimum and maximum values of functions *(p. 53)*, and the average rate of change of a function *(p. 54)*.	To determine whether a function is increasing, decreasing, or constant on an interval, determine whether the graph of the function rises, falls, or is constant from left to right. The points at which the behavior of a function changes can help determine relative minimum or relative maximum values. The average rate of change between any two points is the slope of the line (secant line) through the two points.	59–64
	Identify even and odd functions *(p. 55)*.	**Even:** For each x in the domain of f, $f(-x) = f(x)$. **Odd:** For each x in the domain of f, $f(-x) = -f(x)$.	65, 66
Section 1.6	Identify and graph different types of functions *(pp. 60, 62–64)*, and recognize graphs of parent functions *(p. 64)*.	**Linear:** $f(x) = ax + b$; **Squaring:** $f(x) = x^2$; **Cubic:** $f(x) = x^3$; **Square Root:** $f(x) = \sqrt{x}$; **Reciprocal:** $f(x) = 1/x$ Eight of the most commonly used functions in algebra are shown on page 64.	67–70
Section 1.7	Use vertical and horizontal shifts *(p. 67)*, reflections *(p. 69)*, and nonrigid transformations *(p. 71)* to sketch graphs of functions.	**Vertical shifts:** $h(x) = f(x) + c$ or $h(x) = f(x) - c$ **Horizontal shifts:** $h(x) = f(x - c)$ or $h(x) = f(x + c)$ **Reflection in x-axis:** $h(x) = -f(x)$ **Reflection in y-axis:** $h(x) = f(-x)$ **Nonrigid transformations:** $h(x) = cf(x)$ or $h(x) = f(cx)$	71–80
Section 1.8	Add, subtract, multiply, and divide functions *(p. 76)*, find compositions of functions *(p. 78)*, and use combinations and compositions of functions to model and solve real-life problems *(p. 80)*.	$(f + g)(x) = f(x) + g(x)$ $(f - g)(x) = f(x) - g(x)$ $(fg)(x) = f(x) \cdot g(x)$ $(f/g)(x) = f(x)/g(x), g(x) \neq 0$ The composition of the function f with the function g is $(f \circ g)(x) = f(g(x))$. A composite function can be used to represent the number of bacteria in food as a function of the amount of time the food has been out of refrigeration. (See Example 8.)	81–86
Section 1.9	Find inverse functions informally and verify that two functions are inverse functions of each other *(p. 84)*.	Let f and g be two functions such that $f(g(x)) = x$ for every x in the domain of g and $g(f(x)) = x$ for every x in the domain of f. Under these conditions, the function g is the inverse function of the function f.	87, 88
	Use graphs to verify inverse functions *(p. 86)*, use the Horizontal Line Test *(p. 87)*, and find inverse functions algebraically *(p. 88)*.	If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa. In short, the graph of f^{-1} is a reflection of the graph of f in the line $y = x$. To find an inverse function, replace $f(x)$ with y, interchange the roles of x and y, solve for y, and then replace y with $f^{-1}(x)$.	89–94
Section 1.10	Use mathematical models to approximate sets of data points *(p. 93)*, and use the *regression* feature of a graphing utility to find equations of least squares regression lines *(p. 94)*.	To see how well a model fits a set of data, compare the actual values of y with the model values. (See Example 1.) The sum of the squared differences is the sum of the squares of the differences between actual data values and model values. The least squares regression line is the linear model with the least sum of the squared differences.	95
	Write mathematical models for direct variation, direct variation as an nth power, inverse variation, combined variation, and joint variation *(pp. 95–99)*.	**Direct variation:** $y = kx$ for some nonzero constant k. **Direct variation as an nth power:** $y = kx^n$ for some nonzero constant k. **Inverse variation:** $y = k/x$ for some nonzero constant k. **Joint variation:** $z = kxy$ for some nonzero constant k.	96, 97

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

1.1 Plotting Points in the Cartesian Plane **In Exercises 1 and 2, plot the points.**

1. $(5, 5), (-2, 0), (-3, 6), (-1, -7)$
2. $(0, 6), (8, 1), (5, -4), (-3, -3)$

Determining Quadrant(s) for a Point **In Exercises 3 and 4, determine the quadrant(s) in which (x, y) could be located.**

3. $x > 0$ and $y = -2$
4. $xy = 4$

5. **Plotting, Distance, and Midpoint** Plot the points $(-2, 6)$ and $(4, -3)$. Then find the distance between the points and the midpoint of the line segment joining the points.
6. **Sales** Barnes & Noble had annual sales of \$6.8 billion in 2013 and \$6.1 billion in 2015. Use the Midpoint Formula to estimate the sales in 2014. Assume that the annual sales follow a linear pattern. *(Source: Barnes & Noble, Inc.)*

1.2 Sketching the Graph of an Equation **In Exercises 7–10, construct a table of values that consists of several points of the equation. Use the resulting solution points to sketch the graph of the equation.**

7. $y = 3x - 5$
8. $y = -\frac{1}{2}x + 2$
9. $y = x^2 - 3x$
10. $y = 2x^2 - x - 9$

Finding *x*- and *y*-Intercepts **In Exercises 11–14, find the *x*- and *y*-intercepts of the graph of the equation.**

11. $y = 2x + 7$
12. $y = |x + 1| - 3$
13. $y = (x - 3)^2 - 4$
14. $y = x\sqrt{4 - x^2}$

Intercepts, Symmetry, and Graphing **In Exercises 15–22, find any intercepts and test for symmetry. Then sketch the graph of the equation.**

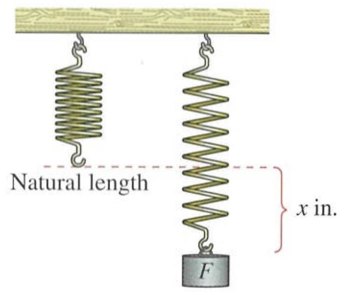
15. $y = -4x + 1$
16. $y = 5x - 6$
17. $y = 6 - x^2$
18. $y = x^2 - 12$
19. $y = x^3 + 5$
20. $y = -6 - x^3$
21. $y = \sqrt{x + 5}$
22. $y = |x| + 9$

Sketching a Circle **In Exercises 23–26, find the center and radius of the circle with the given equation. Then sketch the circle.**

23. $x^2 + y^2 = 9$
24. $x^2 + y^2 = 4$
25. $(x + 2)^2 + y^2 = 16$
26. $x^2 + (y - 8)^2 = 81$

27. **Writing the Equation of a Circle** Write the standard form of the equation of the circle for which the endpoints of a diameter are $(0, 0)$ and $(4, -6)$.
28. **Physics** The force F (in pounds) required to stretch a spring x inches from its natural length (see figure) is

$$F = \frac{5}{4}x, \quad 0 \le x \le 20.$$



(a) Use the model to complete the table.

x	0	4	8	12	16	20
Force, F						

(b) Sketch a graph of the model.

(c) Use the graph to estimate the force necessary to stretch the spring 10 inches.

1.3 Graphing a Linear Equation **In Exercises 29–32, find the slope and *y*-intercept (if possible) of the line. Sketch the line.**

29. $y = -\frac{1}{2}x + 1$
30. $2x - 3y = 6$
31. $y = 1$
32. $x = -6$

Finding the Slope of a Line Through Two Points **In Exercises 33 and 34, find the slope of the line passing through the pair of points.**

33. $(5, -2), (-1, 4)$
34. $(-1, 6), (3, -2)$

Using the Point-Slope Form **In Exercises 35 and 36, find the slope-intercept form of the equation of the line that has the given slope and passes through the given point. Sketch the line.**

35. $m = \frac{1}{3}, \quad (6, -5)$
36. $m = -\frac{3}{4}, \quad (-4, -2)$

Finding an Equation of a Line **In Exercises 37 and 38, find an equation of the line passing through the pair of points. Sketch the line.**

37. $(-6, 4), (4, 9)$
38. $(-9, -3), (-3, -5)$

Finding Parallel and Perpendicular Lines **In Exercises 39 and 40, find equations of the lines that pass through the given point and are (a) parallel to and (b) perpendicular to the given line.**

39. $5x - 4y = 8$, $(3, -2)$
40. $2x + 3y = 5$, $(-8, 3)$

41. **Sales** A discount outlet offers a 20% discount on all items. Write a linear equation giving the sale price S for an item with a list price L.

42. **Hourly Wage** A manuscript translator charges a starting fee of \$50 plus \$2.50 per page translated. Write a linear equation for the amount A earned for translating p pages.

1.4 Testing for Functions Represented Algebraically **In Exercises 43–46, determine whether the equation represents y as a function of x.**

43. $16x - y^4 = 0$
44. $2x - y - 3 = 0$
45. $y = \sqrt{1 - x}$
46. $|y| = x + 2$

Evaluating a Function **In Exercises 47 and 48, find each function value.**

47. $f(x) = x^2 + 1$
(a) $f(2)$
(b) $f(-4)$
(c) $f(t^2)$
(d) $f(t + 1)$

48. $h(x) = |x - 2|$
(a) $h(-4)$
(b) $h(-2)$
(c) $h(0)$
(d) $h(-x + 2)$

Finding the Domain of a Function **In Exercises 49 and 50, find the domain of the function.**

49. $f(x) = \sqrt{25 - x^2}$

50. $h(x) = \dfrac{x}{x^2 - x - 6}$

Physics **In Exercises 51 and 52, the velocity of a ball projected upward from ground level is given by $v(t) = -32t + 48$, where t is the time in seconds and v is the velocity in feet per second.**

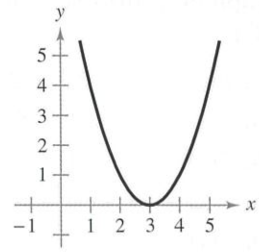
51. Find the velocity when $t = 1$.
52. Find the time when the ball reaches its maximum height. [*Hint:* Find the time when $v(t) = 0$.]

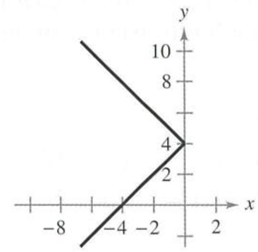
Evaluating a Difference Quotient **In Exercises 53 and 54, find the difference quotient and simplify your answer.**

53. $f(x) = 2x^2 + 3x - 1$, $\dfrac{f(x + h) - f(x)}{h}$, $h \neq 0$

54. $f(x) = x^3 - 5x^2 + x$, $\dfrac{f(x + h) - f(x)}{h}$, $h \neq 0$

1.5 Vertical Line Test for Functions **In Exercises 55 and 56, use the Vertical Line Test to determine whether the graph represents y as a function of x. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

55.


56.


Finding the Zeros of a Function **In Exercises 57 and 58, find the zeros of the function algebraically.**

57. $f(x) = 3x^2 - 16x + 21$
58. $f(x) = 5x^2 + 4x - 1$

Describing Function Behavior **In Exercises 59 and 60, use a graphing utility to graph the function and visually determine the open intervals on which the function is increasing, decreasing, or constant.**

59. $f(x) = |x| + |x + 1|$
60. $f(x) = (x^2 - 4)^2$

Approximating Relative Minima or Maxima **In Exercises 61 and 62, use a graphing utility to approximate (to two decimal places) any relative minima or maxima of the function.**

61. $f(x) = -x^2 + 2x + 1$
62. $f(x) = x^3 - 4x^2 - 1$

Average Rate of Change of a Function **In Exercises 63 and 64, find the average rate of change of the function from x_1 to x_2.**

63. $f(x) = -x^2 + 8x - 4$, $x_1 = 0, x_2 = 4$
64. $f(x) = x^3 + 2x + 1$, $x_1 = 1, x_2 = 3$

Even, Odd, or Neither? **In Exercises 65 and 66, determine whether the function is even, odd, or neither. Then describe the symmetry.**

65. $f(x) = x^4 - 20x^2$
66. $f(x) = 2x\sqrt{x^2 + 3}$

1.6 Writing a Linear Function **In Exercises 67 and 68, (a) write the linear function f that has the given function values and (b) sketch the graph of the function.**

67. $f(2) = -6$, $f(-1) = 3$
68. $f(0) = -5$, $f(4) = -8$

Graphing a Function **In Exercises 69 and 70, sketch the graph of the function.**

69. $g(x) = [\![x]\!] - 2$

70. $f(x) = \begin{cases} 5x - 3, & x \geq -1 \\ -4x + 5, & x < -1 \end{cases}$

1.7 Describing Transformations **In Exercises 71–80, h is related to one of the parent functions described in this chapter. (a) Identify the parent function f. (b) Describe the sequence of transformations from f to h. (c) Sketch the graph of h. (d) Use function notation to write h in terms of f.**

71. $h(x) = x^2 - 9$
72. $h(x) = (x - 2)^3 + 2$
73. $h(x) = -\sqrt{x} + 4$
74. $h(x) = |x + 3| - 5$
75. $h(x) = -(x + 2)^2 + 3$
76. $h(x) = \frac{1}{2}(x - 1)^2 - 2$
77. $h(x) = -[\![x]\!] + 6$
78. $h(x) = -\sqrt{x + 1} + 9$
79. $h(x) = 5[\![x - 9]\!]$
80. $h(x) = -\frac{1}{3}x^3$

1.8 Finding Arithmetic Combinations of Functions **In Exercises 81 and 82, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?**

81. $f(x) = x^2 + 3, \quad g(x) = 2x - 1$
82. $f(x) = x^2 - 4, \quad g(x) = \sqrt{3 - x}$

Finding Domains of Functions and Composite Functions **In Exercises 83 and 84, find (a) $f \circ g$ and (b) $g \circ f$. Find the domain of each function and of each composite function.**

83. $f(x) = \frac{1}{3}x - 3, \quad g(x) = 3x + 1$
84. $f(x) = x^3 - 4, \quad g(x) = \sqrt[3]{x + 7}$

Retail **In Exercises 85 and 86, the price of a washing machine is x dollars. The function**

$$f(x) = x - 100$$

gives the price of the washing machine after a \$100 rebate. The function

$$g(x) = 0.95x$$

gives the price of the washing machine after a 5% discount.

85. Find and interpret $(f \circ g)(x)$.
86. Find and interpret $(g \circ f)(x)$.

1.9 Finding an Inverse Function Informally **In Exercises 87 and 88, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.**

87. $f(x) = 3x + 8$
88. $f(x) = \dfrac{x - 4}{5}$

Applying the Horizontal Line Test **In Exercises 89 and 90, use a graphing utility to graph the function, and use the Horizontal Line Test to determine whether the function has an inverse function.**

89. $f(x) = (x - 1)^2$
90. $h(t) = \dfrac{2}{t - 3}$

Finding and Analyzing Inverse Functions **In Exercises 91 and 92, (a) find the inverse function of f, (b) graph both f and f^{-1} on the same set of coordinate axes, (c) describe the relationship between the graphs of f and f^{-1}, and (d) state the domains and ranges of f and f^{-1}.**

91. $f(x) = \frac{1}{2}x - 3$
92. $f(x) = \sqrt{x + 1}$

Restricting the Domain **In Exercises 93 and 94, restrict the domain of the function f to an interval on which the function is increasing, and find f^{-1} on that interval.**

93. $f(x) = 2(x - 4)^2$
94. $f(x) = |x - 2|$

1.10

95. **Agriculture** The ordered pairs below give the amount B (in millions of pounds) of beef produced on private farms each year from 2007 through 2014. *(Spreadsheet at LarsonPrecalculus.com) (Source: United States Department of Agriculture)*

DATA

(2007, 102.7)	(2010, 84.2)	(2013, 70.4)
(2008, 95.9)	(2011, 75.0)	(2014, 67.9)
(2009, 90.2)	(2012, 76.3)	

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 7$ corresponding to 2007.

(b) Use the *regression* feature of the graphing utility to find the equation of the least squares regression line that fits the data. Then graph the model and the scatter plot you found in part (a) in the same viewing window. How closely does the model represent the data?

96. **Travel Time** The travel time between two cities is inversely proportional to the average speed. A train travels between the cities in 3 hours at an average speed of 65 miles per hour. How long does it take to travel between the cities at an average speed of 80 miles per hour?

97. **Cost** The cost of constructing a wooden box with a square base varies jointly as the height of the box and the square of the width of the box. Constructing a box of height 16 inches and of width 6 inches costs \$28.80. How much does it cost to construct a box of height 14 inches and of width 8 inches?

Exploration

True or False? **In Exercises 98 and 99, determine whether the statement is true or false. Justify your answer.**

98. Relative to the graph of $f(x) = \sqrt{x}$, the graph of the function $h(x) = -\sqrt{x + 9} - 13$ is shifted 9 units to the left and 13 units down, then reflected in the x-axis.

99. If f and g are two inverse functions, then the domain of g is equal to the range of f.

Chapter Test

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

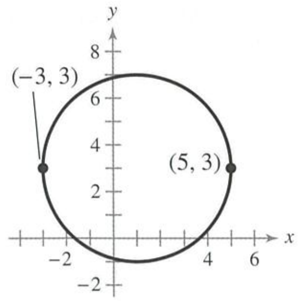


Figure for 6

1. Plot the points $(-2, 5)$ and $(6, 0)$. Then find the distance between the points and the midpoint of the line segment joining the points.
2. A cylindrical can has a radius of 4 centimeters. Write the volume V of the can as a function of the height h.

In Exercises 3–5, find any intercepts and test for symmetry. Then sketch the graph of the equation.

3. $y = 3 - 5x$
4. $y = 4 - |x|$
5. $y = x^2 - 1$

6. Write the standard form of the equation of the circle shown at the left.

In Exercises 7 and 8, find an equation of the line passing through the pair of points. Sketch the line.

7. $(-2, 5), (1, -7)$
8. $(-4, -7), \left(1, \frac{4}{3}\right)$

9. Find equations of the lines that pass through the point $(0, 4)$ and are (a) parallel to and (b) perpendicular to the line $5x + 2y = 3$.
10. Let $f(x) = \dfrac{\sqrt{x + 9}}{x^2 - 81}$. Find (a) $f(7)$, (b) $f(-5)$, and (c) $f(x - 9)$.
11. Find the domain of $f(x) = 10 - \sqrt{3 - x}$.

In Exercises 12–14, (a) find the zeros of the function, (b) use a graphing utility to graph the function, (c) approximate the open intervals on which the function is increasing, decreasing, or constant, and (d) determine whether the function is even, odd, or neither.

12. $f(x) = |x + 5|$
13. $f(x) = 4x\sqrt{3 - x}$
14. $f(x) = 2x^6 + 5x^4 - x^2$

15. Sketch the graph of $f(x) = \begin{cases} 3x + 7, & x \le -3 \\ 4x^2 - 1, & x > -3 \end{cases}$.

In Exercises 16–18, (a) identify the parent function f in the transformation, (b) describe the sequence of transformations from f to h, and (c) sketch the graph of h.

16. $h(x) = 4[\![x]\!]$
17. $h(x) = \sqrt{x + 5} + 8$
18. $h(x) = -2(x - 5)^3 + 3$

In Exercises 19 and 20, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, (d) $(f/g)(x)$, (e) $(f \circ g)(x)$, and (f) $(g \circ f)(x)$.

19. $f(x) = 3x^2 - 7, \quad g(x) = -x^2 - 4x + 5$
20. $f(x) = 1/x, \quad g(x) = 2\sqrt{x}$

In Exercises 21–23, determine whether the function has an inverse function. If it does, find the inverse function.

21. $f(x) = x^3 + 8$
22. $f(x) = |x^2 - 3| + 6$
23. $f(x) = 3x\sqrt{x}$

In Exercises 24–26, find the mathematical model that represents the statement. (Determine the constant of proportionality.)

24. v varies directly as the square root of s. ($v = 24$ when $s = 16$.)
25. A varies jointly as x and y. ($A = 500$ when $x = 15$ and $y = 8$.)
26. b varies inversely as a. ($b = 32$ when $a = 1.5$.)

Proofs in Mathematics

What does the word *proof* mean to you? In mathematics, the word *proof* means a valid argument. When you prove a statement or theorem, you must use facts, definitions, and accepted properties in a logical order. You can also use previously proved theorems in your proof. For example, the proof of the Midpoint Formula below uses the Distance Formula. There are several different proof methods, which you will see in later chapters.

The Midpoint Formula *(p.5)*

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) is

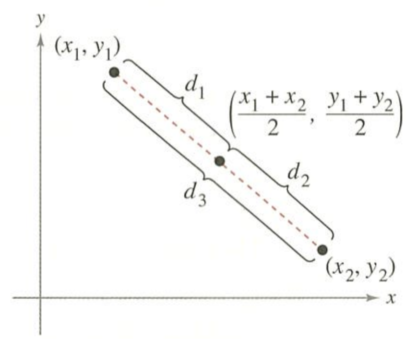
$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

THE CARTESIAN PLANE

The Cartesian plane is named after French mathematician René Descartes (1596–1650). According to some accounts, while Descartes was lying in bed, he noticed a fly buzzing around on the ceiling. He realized that he could describe the fly's position by its distance from the bedroon walls. This led to the development of the Cartesian plane. Descartes felt that using a coordinate plane could facilitate descriptions of the positions of objects.

Proof

Using the figure, you must show that $d_1 = d_2$ and $d_1 + d_2 = d_3$.



By the Distance Formula, you obtain

$$d_1 = \sqrt{\left(\frac{x_1 + x_2}{2} - x_1\right)^2 + \left(\frac{y_1 + y_2}{2} - y_1\right)^2}$$

$$= \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

$$d_2 = \sqrt{\left(x_2 - \frac{x_1 + x_2}{2}\right)^2 + \left(y_2 - \frac{y_1 + y_2}{2}\right)^2}$$

$$= \sqrt{\left(\frac{x_2 - x_1}{2}\right)^2 + \left(\frac{y_2 - y_1}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

and

$$d_3 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

So, it follows that $d_1 = d_2$ and $d_1 + d_2 = d_3$. ■

P.S. Problem Solving

1. **Monthly Wages** As a salesperson, you receive a monthly salary of \$2000, plus a commission of 7% of sales. You receive an offer for a new job at \$2300 per month, plus a commission of 5% of sales.

   (a) Write a linear equation for your current monthly wage W_1 in terms of your monthly sales S.

   (b) Write a linear equation for the monthly wage W_2 of your new job offer in terms of the monthly sales S.

   (c) Use a graphing utility to graph both equations in the same viewing window. Find the point of intersection. What does the point of intersection represent?

   (d) You expect sales of \$20,000 per month. Should you change jobs? Explain.

2. **Cellphone Keypad** For the numbers 2 through 9 on a cellphone keypad (see figure), consider two relations: one mapping numbers onto letters, and the other mapping letters onto numbers. Are both relations functions? Explain.

3. **Sums and Differences of Functions** What can be said about the sum and difference of each pair of functions?

   (a) Two even functions

   (b) Two odd functions

   (c) An odd function and an even function

4. **Inverse Functions** The functions

   $f(x) = x$ and $g(x) = -x$

   are their own inverse functions. Graph each function and explain why this is true. Graph other linear functions that are their own inverse functions. Find a formula for a family of linear functions that are their own inverse functions.

5. **Proof** Prove that a function of the form

   $y = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdots + a_2x^2 + a_0$

   is an even function.

6. **Miniature Golf** A golfer is trying to make a hole-in-one on the miniature golf green shown. The golf ball is at the point (2.5, 2) and the hole is at the point (9.5, 2). The golfer wants to bank the ball off the side wall of the green at the point (x, y). Find the coordinates of the point (x, y). Then write an equation for the path of the ball.

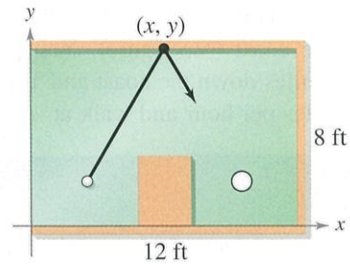


Figure for 6

7. **Titanic** At 2:00 P.M. on April 11, 1912, the *Titanic* left Cobh, Ireland, on her voyage to New York City. At 11:40 P.M. on April 14, the *Titanic* struck an iceberg and sank, having covered only about 2100 miles of the approximately 3400-mile trip.

   (a) What was the total duration of the voyage in hours?

   (b) What was the average speed in miles per hour?

   (c) Write a function relating the distance of the *Titanic* from New York City and the number of hours traveled. Find the domain and range of the function.

   (d) Graph the function in part (c).

8. **Average Rate of Change** Consider the function $f(x) = -x^2 + 4x - 3$. Find the average rate of change of the function from x_1 to x_2.

   (a) $x_1 = 1, x_2 = 2$

   (b) $x_1 = 1, x_2 = 1.5$

   (c) $x_1 = 1, x_2 = 1.25$

   (d) $x_1 = 1, x_2 = 1.125$

   (e) $x_1 = 1, x_2 = 1.0625$

   (f) Does the average rate of change seem to be approaching one value? If so, state the value.

   (g) Find the equations of the secant lines through the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ for parts (a)–(e).

   (h) Find the equation of the line through the point $(1, f(1))$ using your answer from part (f) as the slope of the line.

9. **Inverse of a Composition** Consider the functions $f(x) = 4x$ and $g(x) = x + 6$.

   (a) Find $(f \circ g)(x)$.

   (b) Find $(f \circ g)^{-1}(x)$.

   (c) Find $f^{-1}(x)$ and $g^{-1}(x)$.

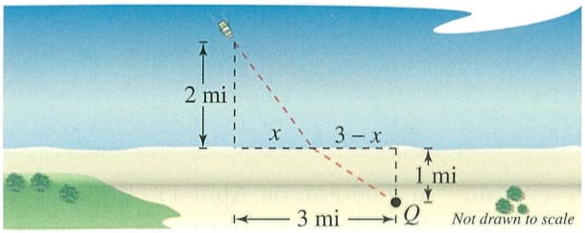
   (d) Find $(g^{-1} \circ f^{-1})(x)$ and compare the result with that of part (b).

   (e) Repeat parts (a) through (d) for $f(x) = x^3 + 1$ and $g(x) = 2x$.

   (f) Write two one-to-one functions f and g, and repeat parts (a) through (d) for these functions.

   (g) Make a conjecture about $(f \circ g)^{-1}(x)$ and $(g^{-1} \circ f^{-1})(x)$.

10. **Trip Time** You are in a boat 2 miles from the nearest point on the coast (see figure). You plan to travel to point Q, 3 miles down the coast and 1 mile inland. You row at 2 miles per hour and walk at 4 miles per hour.



(a) Write the total time T (in hours) of the trip as a function of the distance x (in miles).

(b) Determine the domain of the function.

(c) Use a graphing utility to graph the function. Be sure to choose an appropriate viewing window.

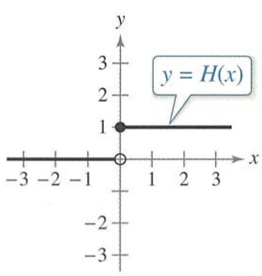
(d) Find the value of x that minimizes T.

(e) Write a brief paragraph interpreting these values.

11. **Heaviside Function** The **Heaviside function**

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

is widely used in engineering applications. (See figure.) To print an enlarged copy of the graph, go to *MathGraphs.com*.



Sketch the graph of each function by hand.

(a) $H(x) - 2$

(b) $H(x - 2)$

(c) $-H(x)$

(d) $H(-x)$

(e) $\frac{1}{2}H(x)$

(f) $-H(x - 2) + 2$

12. **Repeated Composition** Let $f(x) = \dfrac{1}{1 - x}$.

(a) Find the domain and range of f.

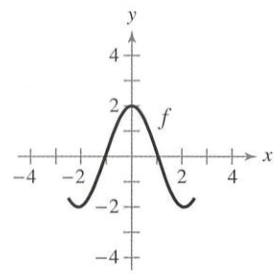
(b) Find $f(f(x))$. What is the domain of this function?

(c) Find $f(f(f(x)))$. Is the graph a line? Why or why not?

13. **Associative Property with Compositions** Show that the Associative Property holds for compositions of functions—that is,

$$(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x).$$

14. **Graphical Reasoning** Use the graph of the function f to sketch the graph of each function. To print an enlarged copy of the graph, go to *MathGraphs.com*.



(a) $f(x + 1)$

(b) $f(x) + 1$

(c) $2f(x)$

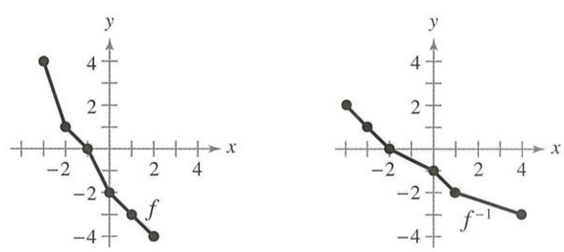
(d) $f(-x)$

(e) $-f(x)$

(f) $|f(x)|$

(g) $f(|x|)$

15. **Graphical Reasoning** Use the graphs of f and f^{-1} to complete each table of function values.



(a)

x	-4	-2	0	4
$(f(f^{-1}(x)))$				

(b)

x	-3	-2	0	1
$(f + f^{-1})(x)$				

(c)

x	-3	-2	0	1
$(f \cdot f^{-1})(x)$				

(d)

x	-4	-3	0	4
$\lvert f^{-1}(x)\rvert$				