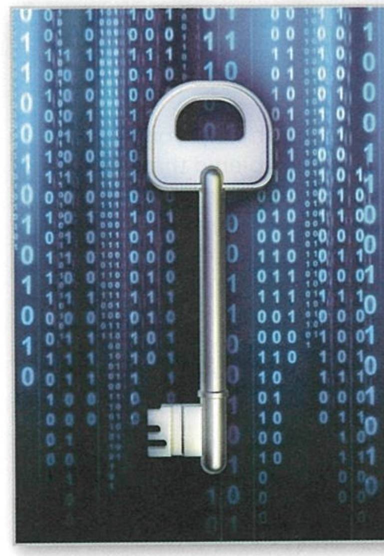
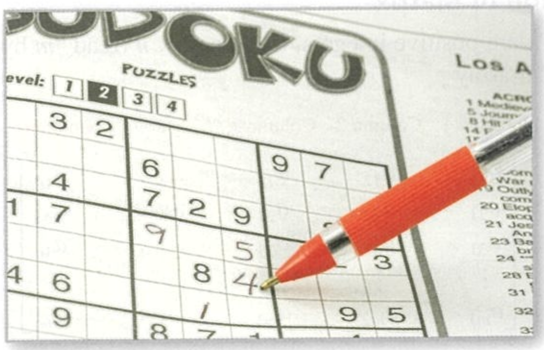
8 Matrices and Determinants



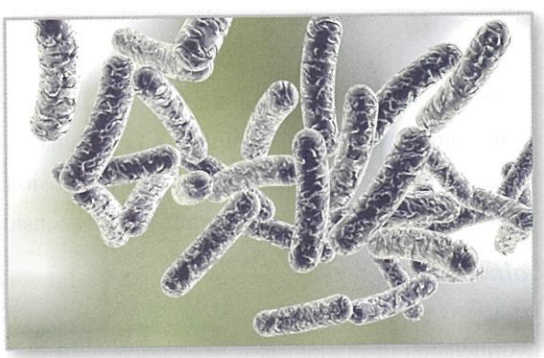
Data Encryption (page 592)



Sudoku (page 581)

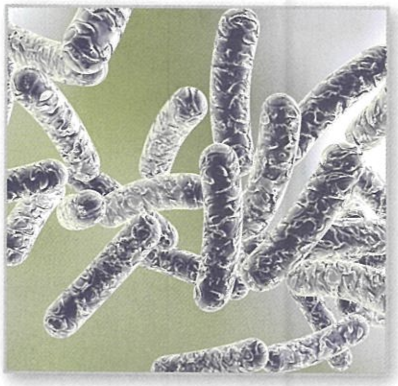
Beam Deflection (page 569)

Flight Crew Scheduling (page 562)



Waterborne Disease (Exercise 93, page 552)

8.1 Matrices and Systems of Equations



Matrices can help you solve real-life problems that are represented by systems of equations. For example, in Exercise 93 on page 552, you will use a matrix to find a model for the numbers of new cases of a waterborne disease in a small city.

- **Write matrices and determine their dimensions.**
- **Perform elementary row operations on matrices.**
- **Use matrices and Gaussian elimination to solve systems of linear equations.**
- **Use matrices and Gauss-Jordan elimination to solve systems of linear equations.**

Matrices

In this section, you will study a streamlined technique for solving systems of linear equations. This technique involves the use of a rectangular array of numbers called a **matrix.** The plural of matrix is *matrices.*

Definition of Matrix

If m and n are positive integers, then an $m \times n$ (read "m by n") matrix is a rectangular array

$$\begin{array}{c} \\ \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \vdots \\ \text{Row } m \end{array}
\begin{array}{c} \begin{array}{ccccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & \ldots & \text{Column } n \end{array} \\
\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

in which each **entry** a_{ij} of the matrix is a number. An $m \times n$ matrix has m rows and n columns.

The entry in the ith row and jth column of a matrix is denoted by the *double subscript* notation a_{ij}. For example, a_{23} refers to the entry in the second row, third column. A matrix having m rows and n columns is said to be of **dimension** $m \times n$. If $m = n$, then the matrix is **square** of dimension $m \times m$ (or $n \times n$). For a square matrix, the entries $a_{11}, a_{22}, a_{33}, \ldots$ are the **main diagonal** entries. A matrix with only one row is called a **row matrix,** and a matrix with only one column is called a **column matrix.**

EXAMPLE 1 Dimensions of Matrices

Determine the dimension of each matrix.

a. $[2]$ **b.** $\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$ **c.** $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ **d.** $\begin{bmatrix} 5 & 0 \\ 2 & -2 \\ -7 & 4 \end{bmatrix}$

Solution

a. This matrix has *one* row and *one* column. The dimension of the matrix is 1×1.

b. This matrix has *one* row and *four* columns. The dimension of the matrix is 1×4.

c. This matrix has *two* rows and *two* columns. The dimension of the matrix is 2×2.

d. This matrix has *three* rows and *two* columns. The dimension of the matrix is 3×2.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Determine the dimension of the matrix $\begin{bmatrix} 14 & 7 & 10 \\ -2 & -3 & -8 \end{bmatrix}$.

A matrix derived from a system of linear equations (each written in standard form with the constant term on the right) is the **augmented matrix** of the system. Moreover, the matrix derived from the coefficients of the system (but not including the constant terms) is the **coefficient matrix** of the system.

System: $\begin{cases} x - 4y + 3z = 5 \\ -x + 3y - z = -3 \\ 2x - 4z = 6 \end{cases}$

Augmented matrix: $\begin{bmatrix} 1 & -4 & 3 & \vdots & 5 \\ -1 & 3 & -1 & \vdots & -3 \\ 2 & 0 & -4 & \vdots & 6 \end{bmatrix}$ *Coefficient matrix:* $\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}$

REMARK The vertical dots in an augmented matrix separate the coefficients of the linear system from the constant terms.

Note the use of 0 for the coefficient of the missing y-variable in the third equation, and also note the fourth column of constant terms in the augmented matrix.

When forming either the coefficient matrix or the augmented matrix of a system, you should begin by vertically aligning the variables in the equations and using zeros for the coefficients of the missing variables.

EXAMPLE 2 Writing an Augmented Matrix

Write the augmented matrix for the system of linear equations.

$$\begin{cases} x + 3y - w = 9 \\ -y + 4z + 2w = -2 \\ x - 5z - 6w = 0 \\ 2x + 4y - 3z = 4 \end{cases}$$

What is the dimension of the augmented matrix?

Solution

Begin by rewriting the linear system and aligning the variables.

$$\begin{cases} x + 3y - w = 9 \\ -y + 4z + 2w = -2 \\ x - 5z - 6w = 0 \\ 2x + 4y - 3z = 4 \end{cases}$$

Next, use the coefficients and constant terms as the matrix entries. Include zeros for the coefficients of the missing variables.

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix} \begin{bmatrix} 1 & 3 & 0 & -1 & \vdots & 9 \\ 0 & -1 & 4 & 2 & \vdots & -2 \\ 1 & 0 & -5 & -6 & \vdots & 0 \\ 2 & 4 & -3 & 0 & \vdots & 4 \end{bmatrix}$$

The augmented matrix has four rows and five columns, so it is a 4×5 matrix. The notation R_n is used to designate each row in the matrix. For example, Row 1 is represented by R_1.

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Write the augmented matrix for the system of linear equations. What is the dimension of the augmented matrix?

$$\begin{cases} x + y + z = 2 \\ 2x - y + 3z = -1 \\ -x + 2y - z = 4 \end{cases}$$

Elementary Row Operations

In Section 7.3, you studied three operations that can be used on a system of linear equations to produce an equivalent system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

In matrix terminology, these three operations correspond to **elementary row operations.** An elementary row operation on an augmented matrix of a given system of linear equations produces a new augmented matrix corresponding to a new (but equivalent) system of linear equations. Two matrices are **row-equivalent** when one can be obtained from the other by a sequence of elementary row operations.

REMARK Although elementary row operations are simple to perform, they involve many arithmetic calculations, with many ways to make a mistake. So, get in the habit of noting the elementary row operations performed in each step to make it more convenient to go back and check your work.

Elementary Row Operations

Operation	Notation
1. Interchange two rows.	$R_a \leftrightarrow R_b$
2. Multiply a row by a nonzero constant.	$cR_a \quad (c \neq 0)$
3. Add a multiple of a row to another row.	$cR_a + R_b$

EXAMPLE 3 Elementary Row Operations

a. Interchange the first and second rows of the original matrix.

Original Matrix

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{matrix} R_2 \\ R_1 \\ \end{matrix} \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

b. Multiply the first row of the original matrix by $\frac{1}{2}$.

Original Matrix

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{matrix} \frac{1}{2}R_1 \rightarrow \\ \\ \end{matrix} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

c. Add -2 times the first row of the original matrix to the third row.

Original Matrix

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{matrix} \\ \\ -2R_1 + R_3 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

Note that the elementary row operation is written beside the row that is *changed.*

TECHNOLOGY Most graphing utilities can perform elementary row operations on matrices. Consult the user's guide for your graphing utility for specific keystrokes.

After performing a row operation, the new row-equivalent matrix that is displayed on your graphing utility is stored in the *answer* variable. So, use the *answer* variable and not the original matrix for subsequent row operations.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Identify the elementary row operation performed to obtain the new row-equivalent matrix.

Original Matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 7 \\ 2 & -6 & 14 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & -6 & 14 \end{bmatrix}$$

Gaussian Elimination with Back-Substitution

In Example 3 in Section 7.3, you used Gaussian elimination with back-substitution to solve a system of linear equations. The next example demonstrates the matrix version of Gaussian elimination. The two methods are essentially the same. The basic difference is that with matrices you do not need to keep writing the variables.

EXAMPLE 4 Comparing Linear Systems and Matrix Operations

Linear System

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$

Associated Augmented Matrix

$$\begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ -1 & 3 & 0 & \vdots & -4 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

Add the first equation to the second equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2x - 5y + 5z = 17 \end{cases}$$

Add the first row to the second row: $R_1 + R_2$.

$$\begin{matrix} \\ R_1 + R_2 \rightarrow \\ \\ \end{matrix} \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 2 & -5 & 5 & \vdots & 17 \end{bmatrix}$$

Add -2 times the first equation to the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ -y - z = -1 \end{cases}$$

Add -2 times the first row to the third row: $-2R_1 + R_3$.

$$\begin{matrix} \\ \\ -2R_1 + R_3 \rightarrow \end{matrix} \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & -1 & -1 & \vdots & -1 \end{bmatrix}$$

Add the second equation to the third equation.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ 2z = 4 \end{cases}$$

Add the second row to the third row: $R_2 + R_3$.

$$\begin{matrix} \\ \\ R_2 + R_3 \rightarrow \end{matrix} \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 2 & \vdots & 4 \end{bmatrix}$$

Multiply the third equation by $\frac{1}{2}$.

$$\begin{cases} x - 2y + 3z = 9 \\ y + 3z = 5 \\ z = 2 \end{cases}$$

Multiply the third row by $\frac{1}{2}$: $\frac{1}{2}R_3$.

$$\begin{matrix} \\ \\ \frac{1}{2}R_3 \rightarrow \end{matrix} \begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

At this point, use back-substitution to find x and y.

$y + 3(2) = 5$ Substitute 2 for z.

$y = -1$ Solve for y.

$x - 2(-1) + 3(2) = 9$ Substitute -1 for y and 2 for z.

$x = 1$ Solve for x.

The solution is $(1, -1, 2)$.

REMARK Remember that you should check a solution by substituting the values of x, y, and z into each equation of the original system. For example, check the solution to Example 4 as shown below.

Equation 1:
$1 - 2(-1) + 3(2) = 9$ ✓

Equation 2:
$-1 + 3(-1) = -4$ ✓

Equation 3:
$2(1) - 5(-1) + 5(2) = 17$ ✓

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Compare solving the linear system below to solving it using its associated augmented matrix.

$$\begin{cases} 2x + y - z = -3 \\ 4x - 2y + 2z = -2 \\ -6x + 5y + 4z = 10 \end{cases}$$

The last matrix in Example 4 is in *row-echelon form.* The term *echelon* refers to the stair-step pattern formed by the nonzero entries of the matrix. The row-echelon form and *reduced row-echelon form* of matrices are described below.

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the following properties.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in *row-echelon form* is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1.

It is worth noting that the row-echelon form of a matrix is not unique. That is, two different sequences of elementary row operations may yield different row-echelon forms. The *reduced* row-echelon form of a matrix, however, is unique.

EXAMPLE 5 Row-Echelon Form

Determine whether each matrix is in row-echelon form. If it is, determine whether it is in reduced row-echelon form.

a. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ **f.** $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution The matrices in (a), (c), (d), and (f) are in row-echelon form. The matrices in (d) and (f) are in *reduced* row-echelon form because every column that has a leading 1 has zeros in every position above and below its leading 1. The matrix in (b) is not in row-echelon form because a row of all zeros occurs above a row that is not all zeros. The matrix in (e) is not in row-echelon form because the first nonzero entry in Row 2 is not a leading 1.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Determine whether the matrix is in row-echelon form. If it is, determine whether it is in reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 11 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Every matrix is row-equivalent to a matrix in row-echelon form. For instance, in Example 5, you can change the matrix in part (e) to row-echelon form by multiplying its second row by $\frac{1}{2}$.

Gaussian elimination with back-substitution works well for solving systems of linear equations by hand or with a computer. For this algorithm, the order in which the elementary row operations are performed is important. You should operate from left to right by columns, using elementary row operations to obtain zeros in all entries directly below the leading 1's.

EXAMPLE 6 Gaussian Elimination with Back-Substitution

Solve the system

$$\begin{cases} y + z - 2w = -3 \\ x + 2y - z = 2 \\ 2x + 4y + z - 3w = -2 \\ x - 4y - 7z - w = -19 \end{cases}$$

Solution

$$\begin{bmatrix} 0 & 1 & 1 & -2 & \vdots & -3 \\ 1 & 2 & -1 & 0 & \vdots & 2 \\ 2 & 4 & 1 & -3 & \vdots & -2 \\ 1 & -4 & -7 & -1 & \vdots & -19 \end{bmatrix}$$ Write augmented matrix.

$$\begin{matrix} R_2 \\ R_1 \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 2 & 4 & 1 & -3 & \vdots & -2 \\ 1 & -4 & -7 & -1 & \vdots & -19 \end{bmatrix}$$ Interchange R_1 and R_2 so first column has leading 1 in upper left corner.

$$\begin{matrix} \\ \\ -2R_1 + R_3 \rightarrow \\ -R_1 + R_4 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 3 & -3 & \vdots & -6 \\ 0 & -6 & -6 & -1 & \vdots & -21 \end{bmatrix}$$ Perform operations on R_3 and R_4 so first column has zeros below its leading 1.

$$\begin{matrix} \\ \\ \\ 6R_2 + R_4 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 3 & -3 & \vdots & -6 \\ 0 & 0 & 0 & -13 & \vdots & -39 \end{bmatrix}$$ Perform operations on R_4 so second column has zeros below its leading 1.

$$\begin{matrix} \\ \\ \frac{1}{3}R_3 \rightarrow \\ -\frac{1}{13}R_4 \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 0 & \vdots & 2 \\ 0 & 1 & 1 & -2 & \vdots & -3 \\ 0 & 0 & 1 & -1 & \vdots & -2 \\ 0 & 0 & 0 & 1 & \vdots & 3 \end{bmatrix}$$ Perform operations on R_3 and R_4 so third and fourth columns have leading 1's.

The matrix is now in row-echelon form, and the corresponding system is

$$\begin{cases} x + 2y - z = 2 \\ y + z - 2w = -3 \\ z - w = -2 \\ w = 3 \end{cases}$$

Using back-substitution, the solution is $(-1, 2, 1, 3)$.

REMARK Note that the order of the variables in the system of equations is x, y, z, and w. The coordinates of the solution are given in this order.

✓ Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Solve the system

$$\begin{cases} -3x + 5y + 3z = -19 \\ 3x + 4y + 4z = 8 \\ 4x - 8y - 6z = 26 \end{cases}$$

The steps below summarize the procedure used in Example 6.

Gaussian Elimination with Back-Substitution

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to rewrite the augmented matrix in row-echelon form.
3. Write the system of linear equations corresponding to the matrix in row-echelon form and use back-substitution to find the solution.

When solving a system of linear equations, remember that it is possible for the system to have no solution. If, in the elimination process, you obtain a row of all zeros except for the last entry, then the system has no solution, or is *inconsistent*.

EXAMPLE 7 A System with No Solution

Solve the system $\begin{cases} x - y + 2z = 4 \\ x + z = 6 \\ 2x - 3y + 5z = 4 \\ 3x + 2y - z = 1 \end{cases}$.

Solution

$$\begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 1 & 0 & 1 & \vdots & 6 \\ 2 & -3 & 5 & \vdots & 4 \\ 3 & 2 & -1 & \vdots & 1 \end{bmatrix}$$ Write augmented matrix.

$$\begin{matrix} \\ -R_1 + R_2 \rightarrow \\ -2R_1 + R_3 \rightarrow \\ -3R_1 + R_4 \rightarrow \end{matrix} \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & -1 & 1 & \vdots & -4 \\ 0 & 5 & -7 & \vdots & -11 \end{bmatrix}$$ Perform row operations.

$$\begin{matrix} \\ \\ R_2 + R_3 \rightarrow \\ \\ \end{matrix} \begin{bmatrix} 1 & -1 & 2 & \vdots & 4 \\ 0 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & -2 \\ 0 & 5 & -7 & \vdots & -11 \end{bmatrix}$$ Perform row operations.

Note that the third row of this matrix consists entirely of zeros except for the last entry. This means that the original system of linear equations is inconsistent. You can see why this is true by converting back to a system of linear equations.

$$\begin{cases} x - y + 2z = 4 \\ y - z = 2 \\ 0 = -2 \\ 5y - 7z = -11 \end{cases}$$

The third equation is not possible, so the system has no solution.

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Solve the system $\begin{cases} x + y + z = 1 \\ x + 2y + 2z = 2 \\ x - y - z = 1 \end{cases}$.

Gauss-Jordan Elimination

With Gaussian elimination, elementary row operations are applied to a matrix to obtain a (row-equivalent) row-echelon form of the matrix. A second method of elimination, called **Gauss-Jordan elimination,** after Carl Friedrich Gauss and Wilhelm Jordan (1842–1899), continues the reduction process until the *reduced* row-echelon form is obtained. This procedure is demonstrated in Example 8.

TECHNOLOGY For a demonstration of a graphical approach to Gauss-Jordan elimination on a 2×3 matrix, see the program called "Visualizing Row Operations," available at *CengageBrain.com.*

EXAMPLE 8 Gauss-Jordan Elimination

See LarsonPrecalculus.com for an interactive version of this type of example.

Use Gauss-Jordan elimination to solve the system $\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4. \\ 2x - 5y + 5z = 17 \end{cases}$

Solution In Example 4, Gaussian elimination was used to obtain the row-echelon form of the linear system above.

$$\begin{bmatrix} 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

Now, rather than using back-substitution, apply elementary row operations until you obtain zeros above each of the leading 1's.

$$2R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & 9 & \vdots & 19 \\ 0 & 1 & 3 & \vdots & 5 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

Perform operations on R_1 so second column has a zero above its leading 1.

$$\begin{matrix} -9R_3 + R_1 \rightarrow \\ -3R_3 + R_2 \rightarrow \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & \vdots & -1 \\ 0 & 0 & 1 & \vdots & 2 \end{bmatrix}$$

Perform operations on R_1 and R_2 so third column has zeros above its leading 1.

The matrix is now in reduced row-echelon form. Converting back to a system of linear equations, you have

$$\begin{cases} x = 1 \\ y = -1. \\ z = 2 \end{cases}$$

So, the solution is $(1, -1, 2)$.

REMARK The advantage of using Gauss-Jordan elimination to solve a system of linear equations is that the solution of the system is easily found without using back-substitution, as illustrated in Example 8.

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Use Gauss-Jordan elimination to solve the system $\begin{cases} -3x + 7y + 2z = 1 \\ -5x + 3y - 5z = -8. \\ 2x - 2y - 3z = 15 \end{cases}$

The elimination procedures described in this section sometimes result in fractional coefficients. For example, consider the system

$$\begin{cases} 2x - 5y + 5z = 17 \\ 3x - 2y + 3z = 11. \\ -3x + 3y = -6 \end{cases}$$

Multiplying the first row by $\frac{1}{2}$ to produce a leading 1 results in fractional coefficients. You can sometimes avoid fractions by judiciously choosing the order in which you apply elementary row operations.

EXAMPLE 9 A System with an Infinite Number of Solutions

Solve the system $\begin{cases} 2x + 4y - 2z = 0 \\ 3x + 5y \qquad = 1 \end{cases}$.

Solution

$$\begin{bmatrix} 2 & 4 & -2 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix}$$

$$\tfrac{1}{2}R_1 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 3 & 5 & 0 & \vdots & 1 \end{bmatrix}$$

$$-3R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & -1 & 3 & \vdots & 1 \end{bmatrix}$$

$$-R_2 \rightarrow \begin{bmatrix} 1 & 2 & -1 & \vdots & 0 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

$$-2R_2 + R_1 \rightarrow \begin{bmatrix} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & -3 & \vdots & -1 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{cases} x + 5z = 2 \\ y - 3z = -1 \end{cases}.$$

Solving for x and y in terms of z, you have

$$x = -5z + 2 \quad \text{and} \quad y = 3z - 1.$$

To write a solution of the system that does not use any of the three variables of the system, let a represent any real number and let $z = a$. Substitute a for z in the equations for x and y.

$$x = -5z + 2 = -5a + 2 \quad \text{and} \quad y = 3z - 1 = 3a - 1$$

So, the solution set can be written as an ordered triple of the form

$$(-5a + 2, 3a - 1, a)$$

where a is any real number. Remember that a solution set of this form represents an infinite number of solutions. Substitute values for a to obtain a few solutions. Then check each solution in the original system of equations.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Solve the system $\begin{cases} 2x - 6y + 6z = 46 \\ 2x - 3y \qquad = 31 \end{cases}$.

Summarize (Section 8.1)

1. State the definition of a matrix *(page 540)*. For examples of writing matrices and determining their dimensions, see Examples 1 and 2.
2. List the elementary row operations *(page 542)*. For an example of performing elementary row operations, see Example 3.
3. Explain how to use matrices and Gaussian elimination to solve systems of linear equations *(page 543)*. For examples of using Gaussian elimination, see Examples 4, 6, and 7.
4. Explain how to use matrices and Gauss-Jordan elimination to solve systems of linear equations *(page 547)*. For examples of using Gauss-Jordan elimination, see Examples 8 and 9.

8.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. A matrix is ________ when the number of rows equals the number of columns.
2. For a square matrix, the entries $a_{11}, a_{22}, a_{33}, \ldots$ are the ________ ________ entries.
3. A matrix derived from a system of linear equations (each written in standard form with the constant term on the right) is the ________ matrix of the system.
4. A matrix derived from the coefficients of a system of linear equations (but not including the constant terms) is the ________ matrix of the system.
5. Two matrices are ________ when one can be obtained from the other by a sequence of elementary row operations.
6. A matrix in row-echelon form is in ________ ________ ________ when every column that has a leading 1 has zeros in every position above and below its leading 1.

Skills and Applications

Dimension of a Matrix In Exercises 7–14, determine the dimension of the matrix.

7. $\begin{bmatrix} 7 & 0 \end{bmatrix}$

8. $\begin{bmatrix} 5 & -3 & 8 & 7 \end{bmatrix}$

9. $\begin{bmatrix} 2 \\ 36 \\ 3 \end{bmatrix}$

10. $\begin{bmatrix} -3 & 7 & 15 & 0 \\ 0 & 0 & 3 & 3 \\ 1 & 1 & 6 & 7 \end{bmatrix}$

11. $\begin{bmatrix} 33 & 45 \\ -9 & 20 \end{bmatrix}$

12. $\begin{bmatrix} -7 & 6 & 4 \\ 0 & -5 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 6 & -1 \\ 8 & 0 & 3 \\ 3 & -9 & 9 \end{bmatrix}$

14. $\begin{bmatrix} 3 & -1 \\ 4 & 1 \\ -5 & 9 \end{bmatrix}$

Writing an Augmented Matrix In Exercises 15–20, write the augmented matrix for the system of linear equations.

15. $\begin{cases} 2x - y = 7 \\ x + y = 2 \end{cases}$

16. $\begin{cases} 5x + 2y = 13 \\ -3x + 4y = -24 \end{cases}$

17. $\begin{cases} x - y + 2z = 2 \\ 4x - 3y + z = -1 \\ 2x + y = 0 \end{cases}$

18. $\begin{cases} -2x - 4y + z = 13 \\ 6x - 7z = 22 \\ 3x - y + z = 9 \end{cases}$

19. $\begin{cases} 3x - 5y + 2z = 12 \\ 12x - 7z = 10 \end{cases}$

20. $\begin{cases} 9x + y - 3z = 21 \\ -15y + 13z = -8 \end{cases}$

Writing a System of Equations In Exercises 21–26, write the system of linear equations represented by the augmented matrix. (Use variables x, y, z, and w, if applicable.)

21. $\begin{bmatrix} 1 & 1 & \vdots & 3 \\ 5 & -3 & \vdots & -1 \end{bmatrix}$

22. $\begin{bmatrix} 5 & 2 & \vdots & 9 \\ 3 & -8 & \vdots & 0 \end{bmatrix}$

23. $\begin{bmatrix} 2 & 0 & 5 & \vdots & -12 \\ 0 & 1 & -2 & \vdots & 7 \\ 6 & 3 & 0 & \vdots & 2 \end{bmatrix}$

24. $\begin{bmatrix} 4 & -5 & -1 & \vdots & 18 \\ -11 & 0 & 6 & \vdots & 25 \\ 3 & 8 & 0 & \vdots & -29 \end{bmatrix}$

25. $\begin{bmatrix} 9 & 12 & 3 & 0 & \vdots & 0 \\ -2 & 18 & 5 & 2 & \vdots & 10 \\ 1 & 7 & -8 & 0 & \vdots & -4 \\ 3 & 0 & 2 & 0 & \vdots & -10 \end{bmatrix}$

26. $\begin{bmatrix} 6 & 2 & -1 & -5 & \vdots & -25 \\ -1 & 0 & 7 & 3 & \vdots & 7 \\ 4 & -1 & -10 & 6 & \vdots & 23 \\ 0 & 8 & 1 & -11 & \vdots & -21 \end{bmatrix}$

Identifying an Elementary Row Operation In Exercises 27–30, identify the elementary row operation(s) performed to obtain the new row-equivalent matrix.

	Original Matrix	New Row-Equivalent Matrix
27.	$\begin{bmatrix} -2 & 5 & 1 \\ 3 & -1 & -8 \end{bmatrix}$	$\begin{bmatrix} 13 & 0 & -39 \\ 3 & -1 & -8 \end{bmatrix}$
28.	$\begin{bmatrix} 3 & -1 & -4 \\ -4 & 3 & 7 \end{bmatrix}$	$\begin{bmatrix} 3 & -1 & -4 \\ 5 & 0 & -5 \end{bmatrix}$
29.	$\begin{bmatrix} 0 & -1 & -5 & 5 \\ -1 & 3 & -7 & 6 \\ 4 & -5 & 1 & 3 \end{bmatrix}$	$\begin{bmatrix} -1 & 3 & -7 & 6 \\ 0 & -1 & -5 & 5 \\ 0 & 7 & -27 & 27 \end{bmatrix}$
30.	$\begin{bmatrix} -1 & -2 & 3 & -2 \\ 2 & -5 & 1 & -7 \\ 5 & 4 & -7 & 6 \end{bmatrix}$	$\begin{bmatrix} -1 & -2 & 3 & -2 \\ 0 & -9 & 7 & -11 \\ 0 & -6 & 8 & -4 \end{bmatrix}$

Elementary Row Operations In Exercises 31–38, fill in the blank(s) using elementary row operations to form a row-equivalent matrix.

31. $\begin{bmatrix} 3 & 6 & 8 \\ 4 & -3 & 6 \end{bmatrix}$

$\begin{bmatrix} 1 & \square & \frac{8}{3} \\ 4 & -3 & 6 \end{bmatrix}$

32. $\begin{bmatrix} 1 & 4 & 3 \\ 2 & 10 & 5 \end{bmatrix}$

$\begin{bmatrix} 1 & 4 & 3 \\ 0 & \square & -1 \end{bmatrix}$

33. $\begin{bmatrix} 1 & 1 & 1 \\ 5 & -2 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & \square & -1 \end{bmatrix}$

34. $\begin{bmatrix} -3 & 3 & 12 \\ 18 & -8 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & \square \\ 18 & -8 & 4 \end{bmatrix}$

35. $\begin{bmatrix} 1 & 5 & 4 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & \square & \square \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 \end{bmatrix}$

36. $\begin{bmatrix} 1 & 0 & 6 & 1 \\ 0 & -1 & 0 & 7 \\ 0 & 0 & -1 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 6 & 1 \\ 0 & 1 & 0 & \square \\ 0 & 0 & 1 & \square \end{bmatrix}$

37. $\begin{bmatrix} 1 & 1 & 4 & -1 \\ 3 & 8 & 10 & 3 \\ -2 & 1 & 12 & 6 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 5 & \square & \square \\ 0 & 3 & \square & \square \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 4 & -1 \\ 0 & 1 & -\frac{2}{5} & \frac{6}{5} \\ 0 & 3 & \square & \square \end{bmatrix}$

38. $\begin{bmatrix} 2 & 4 & 8 & 3 \\ 1 & -1 & -3 & 2 \\ 2 & 6 & 4 & 9 \end{bmatrix}$

$\begin{bmatrix} 1 & \square & \square & \square \\ 1 & -1 & -3 & 2 \\ 2 & 6 & 4 & 9 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 4 & \frac{3}{2} \\ 0 & \square & -7 & \frac{1}{2} \\ 0 & 2 & \square & \square \end{bmatrix}$

Comparing Linear Systems and Matrix Operations In Exercises 39 and 40, (a) perform the row operations to solve the augmented matrix, (b) write and solve the system of linear equations (in variables x, y, and z, if applicable) represented by the augmented matrix, and (c) compare the two solution methods. Which do you prefer?

39. $\left[\begin{array}{cc:c} -3 & 4 & 22 \\ 6 & -4 & -28 \end{array}\right]$

(i) Add R_2 to R_1.

(ii) Add -2 times R_1 to R_2.

(iii) Multiply R_2 by $-\frac{1}{4}$.

(iv) Multiply R_1 by $\frac{1}{3}$.

40. $\left[\begin{array}{ccc:c} 7 & 13 & 1 & -4 \\ -3 & -5 & -1 & -4 \\ 3 & 6 & 1 & -2 \end{array}\right]$

(i) Add R_2 to R_1.

(ii) Multiply R_1 by $\frac{1}{4}$.

(iii) Add R_3 to R_2.

(iv) Add -3 times R_1 to R_3.

(v) Add -2 times R_2 to R_1.

Row-Echelon Form In Exercises 41–44, determine whether the matrix is in row-echelon form. If it is, determine whether it is in reduced row-echelon form.

41. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

42. $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

43. $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

44. $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Writing a Matrix in Row-Echelon Form In Exercises 45–48, write the matrix in row-echelon form. (Remember that the row-echelon form of a matrix is not unique.)

45. $\begin{bmatrix} 1 & 1 & 0 & 5 \\ -2 & -1 & 2 & -10 \\ 3 & 6 & 7 & 14 \end{bmatrix}$

46. $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 7 & -5 & 14 \\ -2 & -1 & -3 & 8 \end{bmatrix}$

47. $\begin{bmatrix} 1 & -1 & -1 & 1 \\ 5 & -4 & 1 & 8 \\ -6 & 8 & 18 & 0 \end{bmatrix}$

48. $\begin{bmatrix} 1 & -3 & 0 & -7 \\ -3 & 10 & 1 & 23 \\ 4 & -10 & 2 & -24 \end{bmatrix}$

Using a Graphing Utility In Exercises 49–54, use the matrix capabilities of a graphing utility to write the matrix in reduced row-echelon form.

49. $\begin{bmatrix} -1 & 2 & 1 \\ 3 & 4 & 9 \\ 2 & 1 & -2 \end{bmatrix}$

50. $\begin{bmatrix} 1 & 3 & 2 \\ 5 & 15 & 9 \\ 2 & 6 & 10 \end{bmatrix}$

51. $\begin{bmatrix} 1 & 2 & 3 & -5 \\ 1 & 2 & 4 & -9 \\ -2 & -4 & -4 & 3 \\ 4 & 8 & 11 & -14 \end{bmatrix}$

52. $\begin{bmatrix} -2 & 3 & -1 & -2 \\ 4 & -2 & 5 & 8 \\ 1 & 5 & -2 & 0 \\ 3 & 8 & -10 & -30 \end{bmatrix}$

53. $\begin{bmatrix} -3 & 5 & 1 & 12 \\ 1 & -1 & 1 & 4 \end{bmatrix}$

54. $\begin{bmatrix} 5 & 1 & 2 & 4 \\ -1 & 5 & 10 & -32 \end{bmatrix}$

Using Back-Substitution In Exercises 55–58, write the system of linear equations represented by the augmented matrix. Then use back-substitution to solve the system. (Use variables x, y, and z, if applicable.)

55. $\left[\begin{array}{cc:c} 1 & -2 & 4 \\ 0 & 1 & -1 \end{array}\right]$

56. $\left[\begin{array}{cc:c} 1 & 5 & 0 \\ 0 & 1 & 6 \end{array}\right]$

57. $\left[\begin{array}{ccc:c} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$

58. $\left[\begin{array}{ccc:c} 1 & 2 & -2 & -1 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 1 & -3 \end{array}\right]$

Gaussian Elimination with Back-Substitution In Exercises 59–68, use matrices to solve the system of linear equations, if possible. Use Gaussian elimination with back-substitution.

59. $\begin{cases} x + 2y = 7 \\ -x + y = 8 \end{cases}$

60. $\begin{cases} 2x + 6y = 16 \\ 2x + 3y = 7 \end{cases}$

61. $\begin{cases} 3x - 2y = -27 \\ x + 3y = 13 \end{cases}$

62. $\begin{cases} -x + y = 4 \\ 2x - 4y = -34 \end{cases}$

63. $\begin{cases} x + 2y - 3z = -28 \\ 4y + 2z = 0 \\ -x + y - z = -5 \end{cases}$

64. $\begin{cases} 3x - 2y + z = 15 \\ -x + y + 2z = -10 \\ x - y - 4z = 14 \end{cases}$

65. $\begin{cases} -3x + 2y = -22 \\ 3x + 4y = 4 \\ 4x - 8y = 32 \end{cases}$

66. $\begin{cases} x + 2y = 0 \\ x + y = 6 \\ 3x - 2y = 8 \end{cases}$

67. $\begin{cases} 3x + 2y - z + w = 0 \\ x - y + 4z + 2w = 25 \\ -2x + y + 2z - w = 2 \\ x + y + z + w = 6 \end{cases}$

68. $\begin{cases} x - 4y + 3z - 2w = 9 \\ 3x - 2y + z - 4w = -13 \\ -4x + 3y - 2z + w = -4 \\ -2x + y - 4z + 3w = -10 \end{cases}$

Interpreting Reduced Row-Echelon Form In Exercises 69 and 70, an augmented matrix that represents a system of linear equations (in variables x, y, and z, if applicable) has been reduced using Gauss-Jordan elimination. Write the solution represented by the augmented matrix.

69. $\left[\begin{array}{cccc} 1 & 0 & \vdots & 3 \\ 0 & 1 & \vdots & -4 \end{array}\right]$

70. $\left[\begin{array}{ccccc} 1 & 0 & 0 & \vdots & 5 \\ 0 & 1 & 0 & \vdots & -3 \\ 0 & 0 & 1 & \vdots & 0 \end{array}\right]$

Gauss-Jordan Elimination In Exercises 71–78, use matrices to solve the system of linear equations, if possible. Use Gauss-Jordan elimination.

71. $\begin{cases} -2x + 6y = -22 \\ x + 2y = -9 \end{cases}$

72. $\begin{cases} 5x - 5y = -5 \\ -2x - 3y = 7 \end{cases}$

73. $\begin{cases} x + 2y + z = 8 \\ 3x + 7y + 6z = 26 \end{cases}$

74. $\begin{cases} x + y + 4z = 5 \\ 2x + y - z = 9 \end{cases}$

75. $\begin{cases} x - 3z = -2 \\ 3x + y - 2z = 5 \\ 2x + 2y + z = 4 \end{cases}$

76. $\begin{cases} 2x - y + 3z = 24 \\ 2y - z = 14 \\ 7x - 5y = 6 \end{cases}$

77. $\begin{cases} -x + y - z = -14 \\ 2x - y + z = 21 \\ 3x + 2y + z = 19 \end{cases}$

78. $\begin{cases} 2x + 2y - z = 2 \\ x - 3y + z = -28 \\ -x + y = 14 \end{cases}$

Using a Graphing Utility In Exercises 79–84, use the matrix capabilities of a graphing utility to write the augmented matrix corresponding to the system of linear equations in reduced row-echelon form. Then solve the system.

79. $\begin{cases} 3x + 3y + 12z = 6 \\ x + y + 4z = 2 \\ 2x + 5y + 20z = 10 \\ -x + 2y + 8z = 4 \end{cases}$

80. $\begin{cases} 2x + 10y + 2z = 6 \\ x + 5y + 2z = 6 \\ x + 5y + z = 3 \\ -3x - 15y - 3z = -9 \end{cases}$

81. $\begin{cases} 2x + y - z + 2w = -6 \\ 3x + 4y + w = 1 \\ x + 5y + 2z + 6w = -3 \\ 5x + 2y - z - w = 3 \end{cases}$

82. $\begin{cases} x + 2y + 2z + 4w = 11 \\ 3x + 6y + 5z + 12w = 30 \\ x + 3y - 3z + 2w = -5 \\ 6x - y - z + w = -9 \end{cases}$

83. $\begin{cases} x + y + z + w = 0 \\ 2x + 3y + z - 2w = 0 \\ 3x + 5y + z = 0 \end{cases}$

84. $\begin{cases} x + 2y + z + 3w = 0 \\ x - y + w = 0 \\ y - z + 2w = 0 \end{cases}$

85. **Error Analysis** Describe the error.

The matrix

$$\begin{bmatrix} 3 \\ 0 \\ 8 \\ -1 \end{bmatrix}$$

has four rows and one column, so the dimension of the matrix is 1×4.

86. **Error Analysis** Describe the error.

The matrix

$$\begin{bmatrix} 1 & 2 & 7 & 2 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

is in reduced row-echelon form.

Curve Fitting **In Exercises 87–92, use a system of linear equations to find the quadratic function $f(x) = ax^2 + bx + c$ that satisfies the given conditions. Solve the system using matrices.**

87. $f(1) = 1,\ f(2) = -1,\ f(3) = -5$

88. $f(1) = 2,\ f(2) = 9,\ f(3) = 20$

89. $f(-2) = -15,\ f(-1) = 7,\ f(1) = -3$

90. $f(-2) = -3,\ f(1) = -3,\ f(2) = -11$

91. $f(1) = 8,\ f(2) = 13,\ f(3) = 20$

92. $f(1) = 9,\ f(2) = 8,\ f(3) = 5$

93. Waterborne Disease

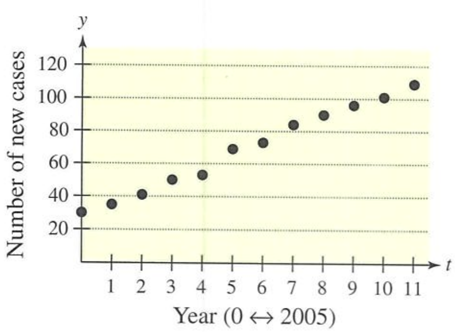
From 2005 through 2016, the numbers of new cases of a waterborne disease in a small city increased in a pattern that was approximately linear (see figure). Find the least squares regression line

$y = at + b$

for the data shown in the figure by solving the system below using matrices. Let t represent the year, with $t = 0$ corresponding to 2005.

$$\begin{cases} 12b + 66a = 831 \\ 66b + 506a = 5643 \end{cases}$$

Use the result to predict the number of new cases of the waterborne disease in 2020. Is the estimate reasonable? Explain.



94. Museum A natural history museum borrows \$2,000,000 at simple annual interest to purchase new exhibits. Some of the money is borrowed at 7%, some at 8.5%, and some at 9.5%. Use a system of linear equations to determine how much is borrowed at each rate given that the total annual interest is \$169,750 and the amount borrowed at 8.5% is four times the amount borrowed at 9.5%. Solve the system of linear equations using matrices.

95. Breeding Facility A city zoo borrows \$2,000,000 at simple annual interest to construct a breeding facility. Some of the money is borrowed at 8%, some at 9%, and some at 12%. Use a system of linear equations to determine how much is borrowed at each rate given that the total annual interest is \$186,000 and the amount borrowed at 8% is twice the amount borrowed at 12%. Solve the system of linear equations using matrices.

96. Mathematical Modeling A video of the path of a ball thrown by a baseball player was analyzed with a grid covering the TV screen. The video was paused three times, and the position of the ball was measured each time. The coordinates obtained are shown in the table. (x and y are measured in feet.)

Horizontal Distance, x	0	15	30
Height, y	5.0	9.6	12.4

(a) Use a system of equations to find the equation of the parabola $y = ax^2 + bx + c$ that passes through the three points. Solve the system using matrices.

(b) Use a graphing utility to graph the parabola.

(c) Graphically approximate the maximum height of the ball and the point at which the ball struck the ground.

(d) Analytically find the maximum height of the ball and the point at which the ball struck the ground.

(e) Compare your results from parts (c) and (d).

Exploration

True or False? **In Exercises 97 and 98, determine whether the statement is true or false. Justify your answer.**

97. $\begin{bmatrix} 5 & 0 & -2 & 7 \\ -1 & 3 & -6 & 0 \end{bmatrix}$ is a 4×2 matrix.

98. The method of Gaussian elimination reduces a matrix until a reduced row-echelon form is obtained.

99. Think About It What is the relationship between the three elementary row operations performed on an augmented matrix and the operations that lead to equivalent systems of equations?

100. HOW DO YOU SEE IT? Determine whether the matrix below is in row-echelon form, reduced row-echelon form, or neither when it satisfies the given conditions.

$$\begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$$

(a) $b = 0, c = 0$ (b) $b \neq 0, c = 0$

(c) $b = 0, c \neq 0$ (d) $b \neq 0, c \neq 0$