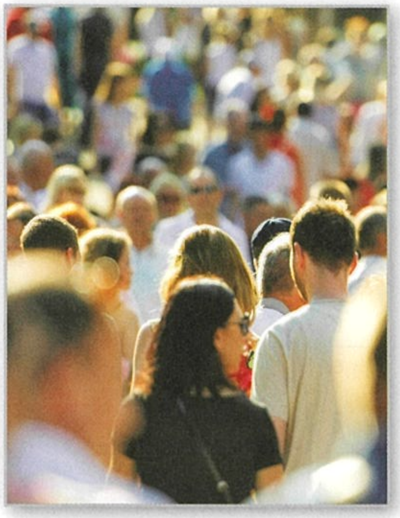
3.5 Exponential and Logarithmic Models



Exponential growth and decay models can often represent populations. For example, in Exercise 30 on page 244, you will use exponential growth and decay models to compare the populations of several countries.

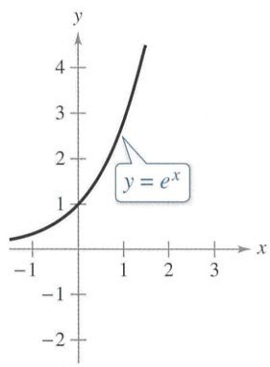
- **Recognize the five most common types of models involving exponential and logarithmic functions.**
- **Use exponential growth and decay functions to model and solve real-life problems.**
- **Use Gaussian functions to model and solve real-life problems.**
- **Use logistic growth functions to model and solve real-life problems.**
- **Use logarithmic functions to model and solve real-life problems.**

Introduction

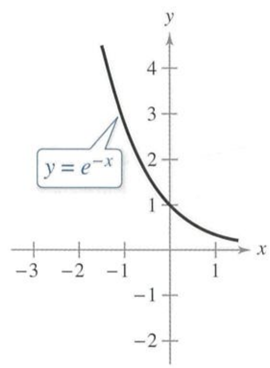
The five most common types of mathematical models involving exponential functions and logarithmic functions are listed below.

1. **Exponential growth model:** $y = ae^{bx}, \quad b > 0$
2. **Exponential decay model:** $y = ae^{-bx}, \quad b > 0$
3. **Gaussian model:** $y = ae^{-(x-b)^2/c}$
4. **Logistic growth model:** $y = \dfrac{a}{1 + be^{-rx}}$
5. **Logarithmic models:** $y = a + b \ln x, \quad y = a + b \log x$

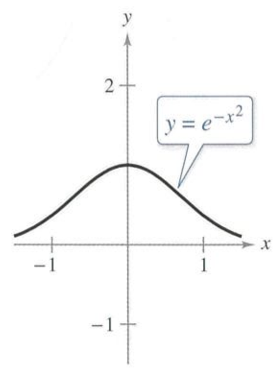
The basic shapes of the graphs of these functions are shown below.



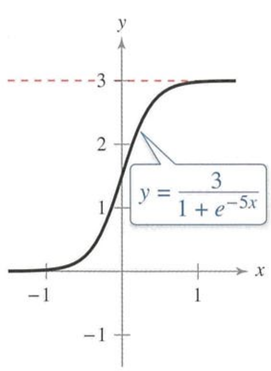
Exponential growth model



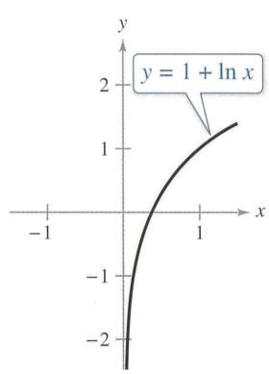
Exponential decay model



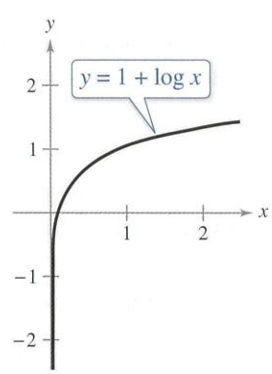
Gaussian model



Logistic growth model



Natural logarithmic model



Common logarithmic model

You often gain insight into a situation modeled by an exponential or logarithmic function by identifying and interpreting the asymptotes of the graph of the function. Identify the asymptote(s) of the graph of each function shown above.

Pavel Vakhrushev/Shutterstock.com

88. Temperature An object at a temperature of 160°C was removed from a furnace and placed in a room at 20°C. The temperature T of the object was measured each hour h and recorded in the table. A model for the data is

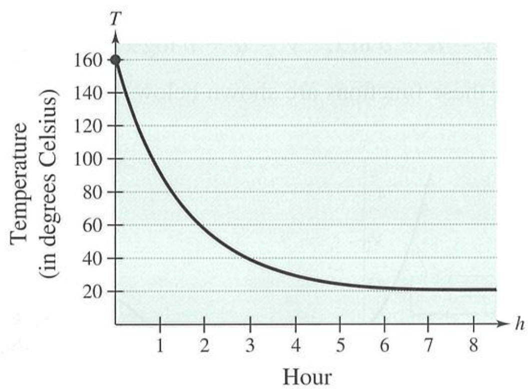
$$T = 20 + 140e^{-0.68h}.$$

DATA

Hour, h	Temperature, T
0	160°
1	90°
2	56°
3	38°
4	29°
5	24°

Spreadsheet at LarsonPrecalculus.com

(a) The figure below shows the graph of the model. Use the graph to identify the horizontal asymptote of the model and interpret the asymptote in the context of the problem.



(b) Use the model to approximate the time it took for the object to reach a temperature of 100°C.

Exploration

True or False? **In Exercises 89–92, rewrite each verbal statement as an equation. Then decide whether the statement is true or false. Justify your answer.**

89. The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.

90. The logarithm of the sum of two numbers is equal to the product of the logarithms of the numbers.

91. The logarithm of the difference of two numbers is equal to the difference of the logarithms of the numbers.

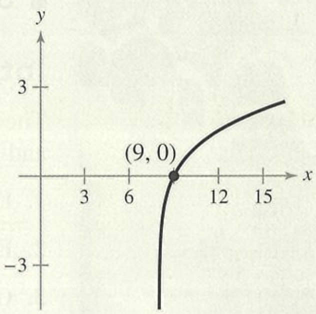
92. The logarithm of the quotient of two numbers is equal to the difference of the logarithms of the numbers.

93. Think About It Is it possible for a logarithmic equation to have more than one extraneous solution? Explain.

94. HOW DO YOU SEE IT? Solving $\log_3 x + \log_3(x - 8) = 2$ algebraically, the solutions appear to be $x = 9$ and $x = -1$. Use the graph of

$$y = \log_3 x + \log_3(x - 8) - 2$$

to determine whether each value is an actual solution of the equation. Explain.



95. Finance You are investing P dollars at an annual interest rate of r, compounded continuously, for t years. Which change below results in the highest value of the investment? Explain.

(a) Double the amount you invest.

(b) Double your interest rate.

(c) Double the number of years.

96. Think About It Are the times required for the investments in Exercises 71 and 72 to quadruple twice as long as the times for them to double? Give a reason for your answer and verify your answer algebraically.

97. Effective Yield The *effective yield* of an investment plan is the percent increase in the balance after 1 year. Find the effective yield for each investment plan. Which investment plan has the greatest effective yield? Which investment plan will have the highest balance after 5 years?

(a) 7% annual interest rate, compounded annually

(b) 7% annual interest rate, compounded continuously

(c) 7% annual interest rate, compounded quarterly

(d) 7.25% annual interest rate, compounded quarterly

98. Graphical Reasoning Let $f(x) = \log_a x$ and $g(x) = a^x$, where $a > 1$.

(a) Let $a = 1.2$ and use a graphing utility to graph the two functions in the same viewing window. What do you observe? Approximate any points of intersection of the two graphs.

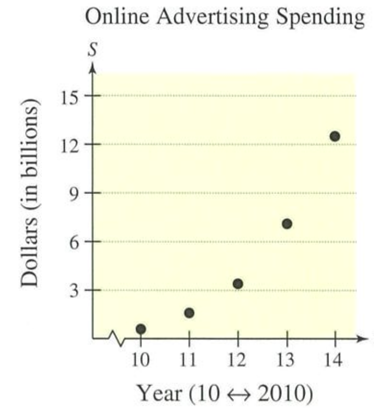
(b) Determine the value(s) of a for which the two graphs have one point of intersection.

(c) Determine the value(s) of a for which the two graphs have two points of intersection.

Exponential Growth and Decay

EXAMPLE 1 Online Advertising

The amounts S (in billions of dollars) spent in the United States on mobile online advertising in the years 2010 through 2014 are shown in the table. A scatter plot of the data is shown at the right. *(Source: IAB/Price Waterhouse Coopers)*



Year	2010	2011	2012	2013	2014
Advertising Spending	0.6	1.6	3.4	7.1	12.5

An exponential growth model that approximates the data is

$$S = 0.00036e^{0.7563t}, \quad 10 \le t \le 14$$

where t represents the year, with $t = 10$ corresponding to 2010. Compare the values found using the model with the amounts shown in the table. According to this model, in what year will the amount spent on mobile online advertising be approximately \$65 billion?

Algebraic Solution

The table compares the actual amounts with the values found using the model.

Year	2010	2011	2012	2013	2014
Advertising Spending	0.6	1.6	3.4	7.1	12.5
Model	0.7	1.5	3.1	6.7	14.3

To find when the amount spent on mobile online advertising is about \$65 billion, let $S = 65$ in the model and solve for t.

$0.00036e^{0.7563t} = S$ Write original model.

$0.00036e^{0.7563t} = 65$ Substitute 65 for S.

$e^{0.7563t} \approx 180{,}556$ Divide each side by 0.00036.

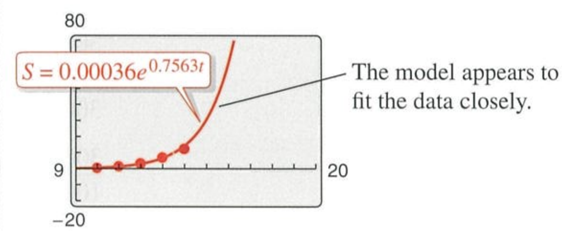
$\ln e^{0.7563t} \approx \ln 180{,}556$ Take natural log of each side.

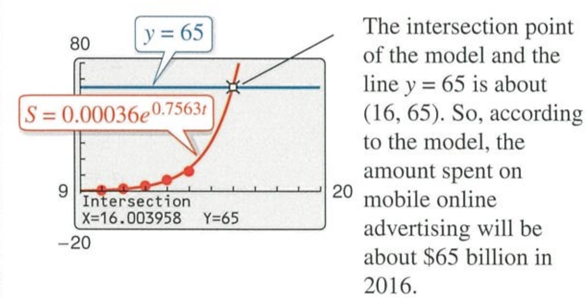
$0.7563t \approx 12.1038$ Inverse Property

$t \approx 16$ Divide each side by 0.7563.

According to the model, the amount spent on mobile online advertising will be about \$65 billion in 2016.

Graphical Solution





Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

In Example 1, in what year will the amount spent on mobile online advertising be about \$300 billion?

TECHNOLOGY Some graphing utilities have an *exponential regression* feature that can help you find exponential models to represent data. If you have such a graphing utility, use it to find an exponential model for the data given in Example 1. How does your model compare with the model given in Example 1?

In Example 1, the exponential growth model is given. Sometimes you must find such a model. One technique for doing this is shown in Example 2.

EXAMPLE 2 Modeling Population Growth

In a research experiment, a population of fruit flies is increasing according to the law of exponential growth. After 2 days there are 100 flies, and after 4 days there are 300 flies. How many flies will there be after 5 days?

Solution Let y be the number of flies at time t (in days). From the given information, you know that $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$. Substituting this information into the model $y = ae^{bt}$ produces

$$100 = ae^{2b} \quad \text{and} \quad 300 = ae^{4b}.$$

To solve for b, solve for a in the first equation.

$$100 = ae^{2b} \qquad \text{Write first equation.}$$

$$\frac{100}{e^{2b}} = a \qquad \text{Solve for } a.$$

Then substitute the result into the second equation.

$$300 = ae^{4b} \qquad \text{Write second equation.}$$

$$300 = \left(\frac{100}{e^{2b}}\right)e^{4b} \qquad \text{Substitute } \frac{100}{e^{2b}} \text{ for } a.$$

$$300 = 100e^{2b} \qquad \text{Simplify.}$$

$$\frac{300}{100} = e^{2b} \qquad \text{Divide each side by 100.}$$

$$\ln 3 = 2b \qquad \text{Take natural log of each side.}$$

$$\frac{1}{2}\ln 3 = b \qquad \text{Solve for } b.$$

Now substitute $\frac{1}{2}\ln 3$ for b in the expression you found for a.

$$a = \frac{100}{e^{2[(1/2)\ln 3]}} \qquad \text{Substitute } \tfrac{1}{2}\ln 3 \text{ for } b.$$

$$= \frac{100}{e^{\ln 3}} \qquad \text{Simplify.}$$

$$= \frac{100}{3} \qquad \text{Inverse Property}$$

$$\approx 33.33 \qquad \text{Divide.}$$

So, with $a \approx 33.33$ and $b = \frac{1}{2}\ln 3 \approx 0.5493$, the exponential growth model is

$$y = 33.33e^{0.5493t}$$

as shown in Figure 3.15. After 5 days, the population will be

$$y = 33.33e^{0.5493(5)}$$

$$\approx 520 \text{ flies.}$$

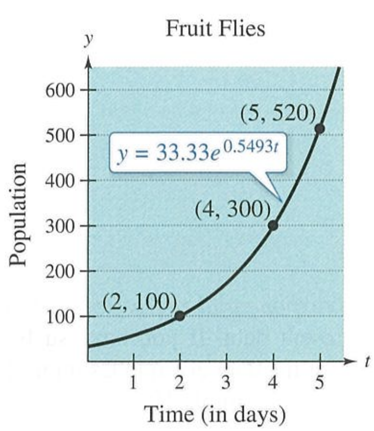
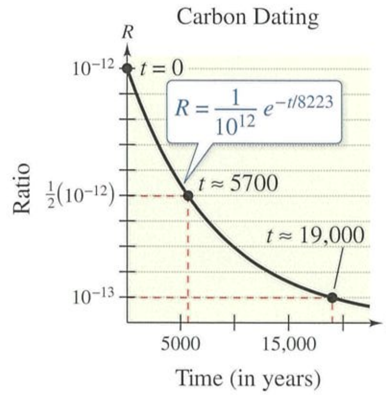


Figure 3.15

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

The number of bacteria in a culture is increasing according to the law of exponential growth. After 1 hour there are 100 bacteria, and after 2 hours there are 200 bacteria. How many bacteria will there be after 3 hours?

In living organic material, the ratio of the number of radioactive carbon isotopes (carbon-14) to the number of nonradioactive carbon isotopes (carbon-12) is about 1 to 10^{12}. When organic material dies, its carbon-12 content remains fixed, whereas its radioactive carbon-14 begins to decay with a half-life of about 5700 years. To estimate the age (the number of years since death) of organic material, scientists use the formula



$$R = \frac{1}{10^{12}}e^{-t/8223} \qquad \text{Carbon dating model}$$

where R represents the ratio of carbon-14 to carbon-12 of organic material t years after death. The graph of R is shown at the right. Note that R decreases as t increases.

EXAMPLE 3 Carbon Dating

Estimate the age of a newly discovered fossil for which the ratio of carbon-14 to carbon-12 is $R = \frac{1}{10^{13}}$.

Algebraic Solution

In the carbon dating model, substitute the given value of R to obtain the following.

$$\frac{1}{10^{12}}e^{-t/8223} = R \qquad \text{Write original model.}$$

$$\frac{e^{-t/8223}}{10^{12}} = \frac{1}{10^{13}} \qquad \text{Substitute } \frac{1}{10^{13}} \text{ for } R.$$

$$e^{-t/8223} = \frac{1}{10} \qquad \text{Multiply each side by } 10^{12}.$$

$$\ln e^{-t/8223} = \ln\frac{1}{10} \qquad \text{Take natural log of each side.}$$

$$-\frac{t}{8223} \approx -2.3026 \qquad \text{Inverse Property}$$

$$t \approx 18{,}934 \qquad \text{Multiply each side by } -8223.$$

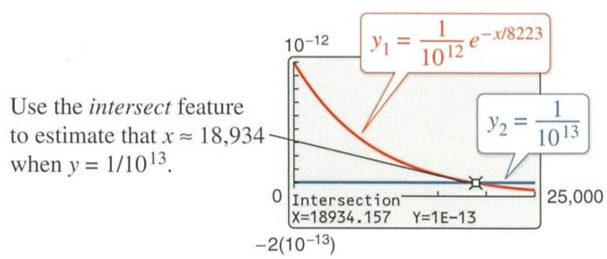
So, to the nearest thousand years, the age of the fossil is about 19,000 years.

Graphical Solution

Use a graphing utility to graph

$$y_1 = \frac{1}{10^{12}}e^{-x/8223} \quad \text{and} \quad y_2 = \frac{1}{10^{13}}$$

in the same viewing window.



So, to the nearest thousand years, the age of the fossil is about 19,000 years.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Estimate the age of a newly discovered fossil for which the ratio of carbon-14 to carbon-12 is $R = 1/10^{14}$.

The value of b in the exponential decay model $y = ae^{-bt}$ determines the *decay* of radioactive isotopes. For example, to find how much of an initial 10 grams of ^{226}Ra isotope with a half-life of 1599 years is left after 500 years, substitute this information into the model $y = ae^{-bt}$.

$$\frac{1}{2}(10) = 10e^{-b(1599)} \implies \ln\frac{1}{2} = -1599b \implies b = -\frac{\ln\frac{1}{2}}{1599}$$

Using the value of b found above and $a = 10$, the amount left is

$$y = 10e^{-[-\ln(1/2)/1599](500)} \approx 8.05 \text{ grams.}$$

Gaussian Models

As mentioned at the beginning of this section, Gaussian models are of the form

$$y = ae^{-(x-b)^2/c}.$$

This type of model is commonly used in probability and statistics to represent populations that are **normally distributed.** For *standard* normal distributions, the model takes the form

$$y = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

The graph of a Gaussian model is called a **bell-shaped curve.** Use a graphing utility to graph the standard normal distribution curve. Can you see why it is called a bell-shaped curve?

The **average value** of a population can be found from the bell-shaped curve by observing where the maximum y-value of the function occurs. The x-value corresponding to the maximum y-value of the function represents the average value of the independent variable—in this case, x.

EXAMPLE 4 SAT Scores

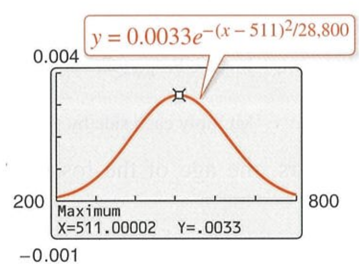
See LarsonPrecalculus.com for an interactive version of this type of example.

In 2015, the SAT mathematics scores for college-bound seniors in the United States roughly followed the normal distribution

$$y = 0.0033e^{-(x-511)^2/28,800}, \quad 200 \le x \le 800$$

where x is the SAT score for mathematics. Use a graphing utility to graph this function and estimate the average SAT mathematics score. *(Source: The College Board)*

Solution The graph of the function is shown below. On this bell-shaped curve, the maximum value of the curve corresponds to the average score. Using the *maximum* feature of the graphing utility, you find that the average mathematics score for college-bound seniors in 2015 was about 511.



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In 2015, the SAT critical reading scores for college-bound seniors in the United States roughly followed the normal distribution

$$y = 0.0034e^{-(x-495)^2/26,912}, \quad 200 \le x \le 800$$

where x is the SAT score for critical reading. Use a graphing utility to graph this function and estimate the average SAT critical reading score. *(Source: The College Board)*

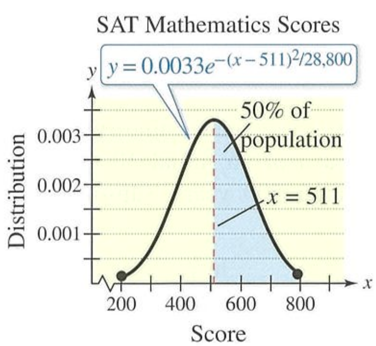


Figure 3.16

In Example 4, note that 50% of the seniors who took the test earned scores greater than 511 (see Figure 3.16).

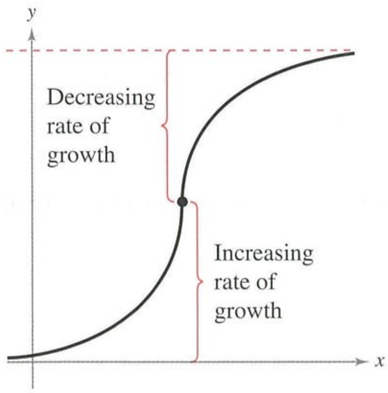


Figure 3.17

Logistic Growth Models

Some populations initially have rapid growth, followed by a declining rate of growth, as illustrated by the graph in Figure 3.17. One model for describing this type of growth pattern is the **logistic curve** given by the function

$$y = \frac{a}{1 + be^{-rx}}$$

where y is the population size and x is the time. An example is a bacteria culture that is initially allowed to grow under ideal conditions and then under less favorable conditions that inhibit growth. A logistic growth curve is also called a **sigmoidal curve.**

EXAMPLE 5 Spread of a Virus

On a college campus of 5000 students, one student returns from vacation with a contagious and long-lasting flu virus. The spread of the virus is modeled by

$$y = \frac{5000}{1 + 4999e^{-0.8t}}, \quad t \geq 0$$

where y is the total number of students infected after t days. The college will cancel classes when 40% or more of the students are infected.

a. How many students are infected after 5 days?

b. After how many days will the college cancel classes?

Algebraic Solution

a. After 5 days, the number of students infected is

$$y = \frac{5000}{1 + 4999e^{-0.8(5)}} = \frac{5000}{1 + 4999e^{-4}} \approx 54.$$

b. The college will cancel classes when the number of infected students is $(0.40)(5000) = 2000$.

$$2000 = \frac{5000}{1 + 4999e^{-0.8t}}$$

$$1 + 4999e^{-0.8t} = 2.5$$

$$e^{-0.8t} = \frac{1.5}{4999}$$

$$-0.8t = \ln\frac{1.5}{4999}$$

$$t = -\frac{1}{0.8}\ln\frac{1.5}{4999}$$

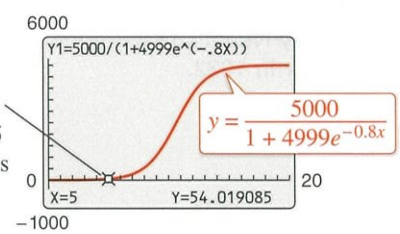
$$t \approx 10.14$$

So, after about 10 days, at least 40% of the students will be infected, and the college will cancel classes.

Graphical Solution

a.

Use the *value* feature to estimate that $y \approx 54$ when $x = 5$. So, after 5 days, about 54 students are infected.

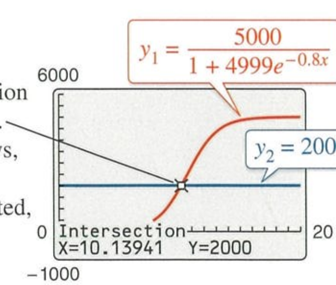


b. The college will cancel classes when the number of infected students is $(0.40)(5000) = 2000$. Use a graphing utility to graph

$$y_1 = \frac{5000}{1 + 4999e^{-0.8x}} \quad \text{and} \quad y_2 = 2000$$

in the same viewing window. Use the *intersect* feature of the graphing utility to find the point of intersection of the graphs.

The point of intersection occurs near $x \approx 10.14$. So, after about 10 days, at least 40% of the students will be infected, and the college will cancel classes.



Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

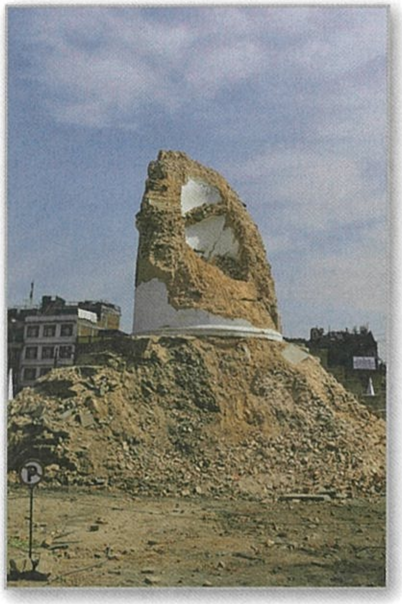
In Example 5, after how many days are 250 students infected?

Logarithmic Models

On the Richter scale, the magnitude R of an earthquake of intensity I is given by

$$R = \log \frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. (Intensity is a measure of the wave energy of an earthquake.)



On April 25, 2015, an earthquake of magnitude 7.8 struck in Nepal. The city of Kathmandu took extensive damage, including the collapse of the 203-foot Dharahara Tower, built by Nepal's first prime minister in 1832.

EXAMPLE 6 Magnitudes of Earthquakes

Find the intensity of each earthquake.

a. Piedmont, California, in 2015: $R = 4.0$ **b.** Nepal in 2015: $R = 7.8$

Solution

a. Because $I_0 = 1$ and $R = 4.0$, you have

$$4.0 = \log \frac{I}{1} \quad \text{Substitute 1 for } I_0 \text{ and 4.0 for } R.$$

$$10^{4.0} = 10^{\log I} \quad \text{Exponentiate each side.}$$

$$10^{4.0} = I \quad \text{Inverse Property}$$

$$10{,}000 = I. \quad \text{Simplify.}$$

b. For $R = 7.8$, you have

$$7.8 = \log \frac{I}{1} \quad \text{Substitute 1 for } I_0 \text{ and 7.8 for } R.$$

$$10^{7.8} = 10^{\log I} \quad \text{Exponentiate each side.}$$

$$10^{7.8} = I \quad \text{Inverse Property}$$

$$63{,}000{,}000 \approx I. \quad \text{Use a calculator.}$$

Note that an increase of 3.8 units on the Richter scale (from 4.0 to 7.8) represents an increase in intensity by a factor of $10^{7.8}/10^4 \approx 63{,}000{,}000/10{,}000 = 6300$. In other words, the intensity of the earthquake in Nepal was about 6300 times as great as that of the earthquake in Piedmont, California.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Find the intensities of earthquakes whose magnitudes are (a) $R = 6.0$ and (b) $R = 7.9$.

Summarize (Section 3.5)

1. State the five most common types of models involving exponential and logarithmic functions *(page 236)*.
2. Describe examples of real-life applications that use exponential growth and decay functions *(pages 237–239, Examples 1–3)*.
3. Describe an example of a real-life application that uses a Gaussian function *(page 240, Example 4)*.
4. Describe an example of a real-life application that uses a logistic growth function *(page 241, Example 5)*.
5. Describe an example of a real-life application that uses a logarithmic function *(page 242, Example 6)*.

3.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. An exponential growth model has the form ________, and an exponential decay model has the form ________.
2. A logarithmic model has the form ________ or ________.
3. In probability and statistics, Gaussian models commonly represent populations that are ________ ________.
4. A logistic growth model has the form ________.

Skills and Applications

Solving for a Variable **In Exercises 5 and 6, (a) solve for P and (b) solve for t.**

5. $A = Pe^{rt}$

6. $A = P\left(1 + \frac{r}{n}\right)^{nt}$

Compound Interest **In Exercises 7–12, find the missing values assuming continuously compounded interest.**

	Initial Investment	Annual % Rate	Time to Double	Amount After 10 Years
7.	\$1000	3.5%		
8.	\$750	$10\frac{1}{2}\%$		
9.	\$750		$7\frac{3}{4}$ yr	
10.	\$500			\$1505.00
11.		4.5%		\$10,000.00
12.			12 yr	\$2000.00

Compound Interest **In Exercises 13 and 14, determine the principal P that must be invested at rate r, compounded monthly, so that \$500,000 will be available for retirement in t years.**

13. $r = 5\%, t = 10$

14. $r = 3\frac{1}{2}\%, t = 15$

Compound Interest **In Exercises 15 and 16, determine the time necessary for P dollars to double when it is invested at interest rate r compounded (a) annually, (b) monthly, (c) daily, and (d) continuously.**

15. $r = 10\%$

16. $r = 6.5\%$

17. **Compound Interest** Complete the table for the time t (in years) necessary for P dollars to triple when it is invested at an interest rate r compounded (a) continuously and (b) annually.

r	2%	4%	6%	8%	10%	12%
t						

18. **Modeling Data** Draw scatter plots of the data in Exercise 17. Use the *regression* feature of a graphing utility to find models for the data.

19. **Comparing Models** If \$1 is invested over a 10-year period, then the balance A after t years is given by either $A = 1 + 0.075[\![t]\!]$ or $A = e^{0.07t}$ depending on whether the interest is simple interest at $7\frac{1}{2}\%$ or continuous compound interest at 7%. Graph each function on the same set of axes. Which grows at a greater rate? (Remember that $[\![t]\!]$ is the greatest integer function discussed in Section 1.6.)

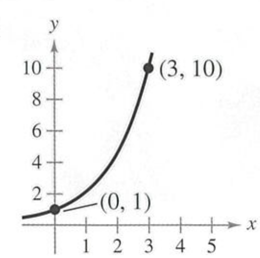
20. **Comparing Models** If \$1 is invested over a 10-year period, then the balance A after t years is given by either $A = 1 + 0.06[\![t]\!]$ or $A = [1 + (0.055/365)]^{[\![365t]\!]}$ depending on whether the interest is simple interest at 6% or compound interest at $5\frac{1}{2}\%$ compounded daily. Use a graphing utility to graph each function in the same viewing window. Which grows at a greater rate?

Radioactive Decay **In Exercises 21–24, find the missing value for the radioactive isotope.**

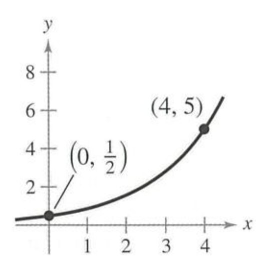
	Isotope	Half-life (years)	Initial Quantity	Amount After 1000 Years
21.	^{226}Ra	1599	10 g	
22.	^{14}C	5715	6.5 g	
23.	^{14}C	5715		2 g
24.	^{239}Pu	24,100		0.4 g

Finding an Exponential Model **In Exercises 25–28, find the exponential model that fits the points shown in the graph or table.**

25.



26.



27.

x	0	4
y	5	1

28.

x	0	3
y	1	$\frac{1}{4}$

29. Population The populations P (in thousands) of Horry County, South Carolina, from 1971 through 2014 can be modeled by

$$P = 76.6e^{0.0313t}$$

where t represents the year, with $t = 1$ corresponding to 1971. *(Source: U.S. Census Bureau)*

(a) Use the model to complete the table.

Year	Population
1980	
1990	
2000	
2010	

(b) According to the model, when will the population of Horry County reach 360,000?

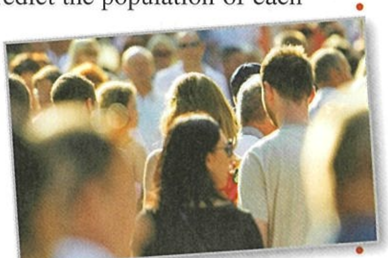
(c) Do you think the model is valid for long-term predictions of the population? Explain.

30. Population

The table shows the mid-year populations (in millions) of five countries in 2015 and the projected populations (in millions) for the year 2025. *(Source: U.S. Census Bureau)*

Country	2015	2025
Bulgaria	7.2	6.7
Canada	35.1	37.6
China	1367.5	1407.0
United Kingdom	64.1	67.2
United States	321.4	347.3

(a) Find the exponential growth or decay model $y = ae^{bt}$ or $y = ae^{-bt}$ for the population of each country by letting $t = 15$ correspond to 2015. Use the model to predict the population of each country in 2035.



(b) You can see that the populations of the United States and the United Kingdom are growing at different rates. What constant in the equation $y = ae^{bt}$ gives the growth rate? Discuss the relationship between the different growth rates and the magnitude of the constant.

31. Website Growth The number y of hits a new website receives each month can be modeled by $y = 4080e^{kt}$, where t represents the number of months the website has been operating. In the website's third month, there were 10,000 hits. Find the value of k, and use this value to predict the number of hits the website will receive after 24 months.

32. Population The population P (in thousands) of Tallahassee, Florida, from 2000 through 2014 can be modeled by $P = 150.9e^{kt}$, where t represents the year, with $t = 0$ corresponding to 2000. In 2005, the population of Tallahassee was about 163,075. *(Source: U.S. Census Bureau)*

(a) Find the value of k. Is the population increasing or decreasing? Explain.

(b) Use the model to predict the populations of Tallahassee in 2020 and 2025. Are the results reasonable? Explain.

(c) According to the model, during what year will the population reach 200,000?

33. Bacteria Growth The number of bacteria in a culture is increasing according to the law of exponential growth. After 3 hours there are 100 bacteria, and after 5 hours there are 400 bacteria. How many bacteria will there be after 6 hours?

34. Bacteria Growth The number of bacteria in a culture is increasing according to the law of exponential growth. The initial population is 250 bacteria, and the population after 10 hours is double the population after 1 hour. How many bacteria will there be after 6 hours?

35. Depreciation A laptop computer that costs $575 new has a book value of $275 after 2 years.

(a) Find the linear model $V = mt + b$.

(b) Find the exponential model $V = ae^{kt}$.

(c) Use a graphing utility to graph the two models in the same viewing window. Which model depreciates faster in the first 2 years?

(d) Find the book values of the computer after 1 year and after 3 years using each model.

(e) Explain the advantages and disadvantages of using each model to a buyer and a seller.

36. Learning Curve The management at a plastics factory has found that the maximum number of units a worker can produce in a day is 30. The learning curve for the number N of units produced per day after a new employee has worked t days is modeled by $N = 30(1 - e^{kt})$. After 20 days on the job, a new employee produces 19 units.

(a) Find the learning curve for this employee. (*Hint:* First, find the value of k.)

(b) How many days does the model predict will pass before this employee is producing 25 units per day?

37. Carbon Dating The ratio of carbon-14 to carbon-12 in a piece of wood discovered in a cave is $R = 1/8^{14}$. Estimate the age of the piece of wood.

38. Carbon Dating The ratio of carbon-14 to carbon-12 in a piece of paper buried in a tomb is $R = 1/13^{11}$. Estimate the age of the piece of paper.

39. IQ Scores The IQ scores for a sample of students at a small college roughly follow the normal distribution

$$y = 0.0266e^{-(x-100)^2/450}, \quad 70 \le x \le 115$$

where x is the IQ score.

(a) Use a graphing utility to graph the function.

(b) From the graph in part (a), estimate the average IQ score of a student.

40. Education The amount of time (in hours per week) a student utilizes a math-tutoring center roughly follows the normal distribution

$$y = 0.7979e^{-(x-5.4)^2/0.5}, \quad 4 \le x \le 7$$

where x is the number of hours.

(a) Use a graphing utility to graph the function.

(b) From the graph in part (a), estimate the average number of hours per week a student uses the tutoring center.

41. Cell Sites A cell site is a site where electronic communications equipment is placed in a cellular network for the use of mobile phones. The numbers y of cell sites from 1985 through 2014 can be modeled by

$$y = \frac{320{,}110}{1 + 374e^{-0.252t}}$$

where t represents the year, with $t = 5$ corresponding to 1985. *(Source: CTIA-The Wireless Association)*

(a) Use the model to find the numbers of cell sites in the years 1998, 2003, and 2006.

(b) Use a graphing utility to graph the function.

(c) Use the graph to determine the year in which the number of cell sites reached 270,000.

(d) Confirm your answer to part (c) algebraically.

42. Population The population P (in thousands) of a city from 2000 through 2016 can be modeled by

$$P = \frac{2632}{1 + 0.083e^{0.050t}}$$

where t represents the year, with $t = 0$ corresponding to 2000.

(a) Use the model to find the populations of the city in the years 2000, 2005, 2010, and 2015.

(b) Use a graphing utility to graph the function.

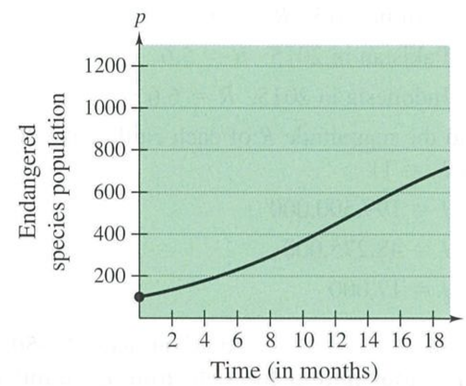
(c) Use the graph to determine the year in which the population reached 2.2 million.

(d) Confirm your answer to part (c) algebraically.

43. Population Growth A conservation organization released 100 animals of an endangered species into a game preserve. The preserve has a carrying capacity of 1000 animals. The growth of the pack is modeled by the logistic curve

$$p(t) = \frac{1000}{1 + 9e^{-0.1656t}}$$

where t is measured in months (see figure).



(a) Estimate the population after 5 months.

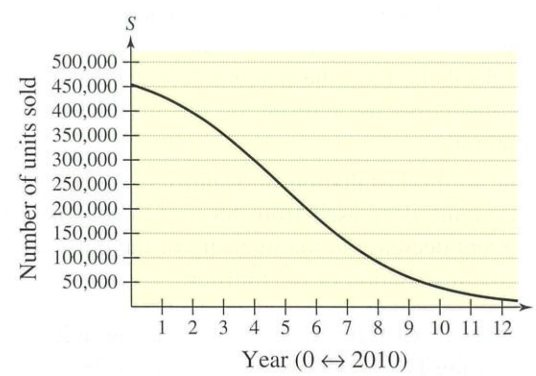
(b) After how many months is the population 500?

(c) Use a graphing utility to graph the function. Use the graph to determine the horizontal asymptotes, and interpret the meaning of the asymptotes in the context of the problem.

44. Sales After discontinuing all advertising for a tool kit in 2010, the manufacturer noted that sales began to drop according to the model

$$S = \frac{500{,}000}{1 + 0.1e^{kt}}$$

where S represents the number of units sold and t represents the year, with $t = 0$ corresponding to 2010 (see figure). In 2014, 300,000 units were sold.



(a) Use the graph to estimate sales in 2020.

(b) Complete the model by solving for k.

(c) Use the model to estimate sales in 2020. Compare your results with that of part (a).

Geology In Exercises 45 and 46, use the Richter scale

$$R = \log \frac{I}{I_0}$$

for measuring the magnitude R of an earthquake.

45. Find the intensity I of an earthquake measuring R on the Richter scale (let $I_0 = 1$).
    (a) Peru in 2015: $R = 7.6$
    (b) Pakistan in 2015: $R = 5.6$
    (c) Indonesia in 2015: $R = 6.6$
46. Find the magnitude R of each earthquake of intensity I (let $I_0 = 1$).
    (a) $I = 199{,}500{,}000$
    (b) $I = 48{,}275{,}000$
    (c) $I = 17{,}000$

Intensity of Sound In Exercises 47–50, use the following information for determining sound intensity. The number of decibels β of a sound with an intensity of I watts per square meter is given by $\beta = 10 \log(I/I_0)$, where I_0 is an intensity of 10^{-12} watt per square meter, corresponding roughly to the faintest sound that can be heard by the human ear. In Exercises 47 and 48, find the number of decibels β of the sound.

47. (a) $I = 10^{-10}$ watt per m^2 (quiet room)
    (b) $I = 10^{-5}$ watt per m^2 (busy street corner)
    (c) $I = 10^{-8}$ watt per m^2 (quiet radio)
    (d) $I = 10^{-3}$ watt per m^2 (loud car horn)
48. (a) $I = 10^{-11}$ watt per m^2 (rustle of leaves)
    (b) $I = 10^{2}$ watt per m^2 (jet at 30 meters)
    (c) $I = 10^{-4}$ watt per m^2 (door slamming)
    (d) $I = 10^{-6}$ watt per m^2 (normal conversation)
49. Due to the installation of noise suppression materials, the noise level in an auditorium decreased from 93 to 80 decibels. Find the percent decrease in the intensity of the noise as a result of the installation of these materials.
50. Due to the installation of a muffler, the noise level of an engine decreased from 88 to 72 decibels. Find the percent decrease in the intensity of the noise as a result of the installation of the muffler.

pH Levels In Exercises 51–56, use the acidity model $\text{pH} = -\log[H^+]$, where acidity (pH) is a measure of the hydrogen ion concentration $[H^+]$ (measured in moles of hydrogen per liter) of a solution.

51. Find the pH when $[H^+] = 2.3 \times 10^{-5}$.
52. Find the pH when $[H^+] = 1.13 \times 10^{-5}$.
53. Compute $[H^+]$ for a solution in which $\text{pH} = 5.8$.
54. Compute $[H^+]$ for a solution in which $\text{pH} = 3.2$.
55. Apple juice has a pH of 2.9 and drinking water has a pH of 8.0. The hydrogen ion concentration of the apple juice is how many times the concentration of drinking water?
56. The pH of a solution decreases by one unit. By what factor does the hydrogen ion concentration increase?
57. **Forensics** At 8:30 A.M., a coroner went to the home of a person who had died during the night. In order to estimate the time of death, the coroner took the person's temperature twice. At 9:00 A.M. the temperature was 85.7°F, and at 11:00 A.M. the temperature was 82.8°F. From these two temperatures, the coroner was able to determine that the time elapsed since death and the body temperature were related by the formula

$$t = -10 \ln \frac{T - 70}{98.6 - 70}$$

where t is the time in hours elapsed since the person died and T is the temperature (in degrees Fahrenheit) of the person's body. (This formula comes from a general cooling principle called *Newton's Law of Cooling*. It uses the assumptions that the person had a normal body temperature of 98.6°F at death and that the room temperature was a constant 70°F.) Use the formula to estimate the time of death of the person.

58. **Home Mortgage** A $120,000 home mortgage for 30 years at $7\frac{1}{2}\%$ has a monthly payment of $839.06. Part of the monthly payment covers the interest charge on the unpaid balance, and the remainder of the payment reduces the principal. The amount paid toward the interest is

$$u = M - \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}$$

and the amount paid toward the reduction of the principal is

$$v = \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}.$$

In these formulas, P is the amount of the mortgage, r is the interest rate (in decimal form), M is the monthly payment, and t is the time in years.

(a) Use a graphing utility to graph each function in the same viewing window. (The viewing window should show all 30 years of mortgage payments.)
(b) In the early years of the mortgage, is the greater part of the monthly payment paid toward the interest or the principal? Approximate the time when the monthly payment is evenly divided between interest and principal reduction.
(c) Repeat parts (a) and (b) for a repayment period of 20 years ($M = \$966.71$). What can you conclude?

59. Home Mortgage The total interest u paid on a home mortgage of P dollars at interest rate r (in decimal form) for t years is

$$u = P\left[\frac{rt}{1 - \left(\frac{1}{1 + r/12}\right)^{12t}} - 1\right].$$

Consider a \$120,000 home mortgage at $7\frac{1}{2}\%$.

(a) Use a graphing utility to graph the total interest function.

(b) Approximate the length of the mortgage for which the total interest paid is the same as the size of the mortgage. Is it possible that some people are paying twice as much in interest charges as the size of the mortgage?

60. Car Speed The table shows the time t (in seconds) required for a car to attain a speed of s miles per hour from a standing start.

DATA

Speed, s	Time, t
30	3.4
40	5.0
50	7.0
60	9.3
70	12.0
80	15.8
90	20.0

Spreadsheet at LarsonPrecalculus.com

Two models for these data are given below.

$t_1 = 40.757 + 0.556s - 15.817 \ln s$

$t_2 = 1.2259 + 0.0023s^2$

(a) Use the *regression* feature of a graphing utility to find a linear model t_3 and an exponential model t_4 for the data.

(b) Use the graphing utility to graph the data and each model in the same viewing window.

(c) Create a table comparing the data with estimates obtained from each model.

(d) Use the results of part (c) to find the sum of the absolute values of the differences between the data and the estimated values found using each model. Based on the four sums, which model do you think best fits the data? Explain.

Exploration

True or False? **In Exercises 61–64, determine whether the statement is true or false. Justify your answer.**

61. The domain of a logistic growth function cannot be the set of real numbers.

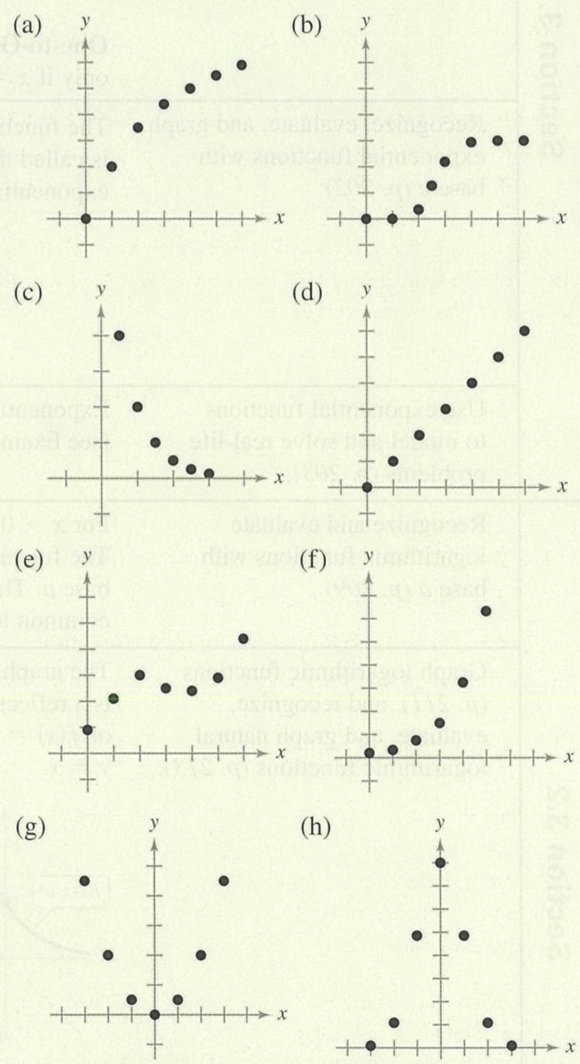
62. A logistic growth function will always have an x-intercept.

63. The graph of $f(x) = \dfrac{4}{1 + 6e^{-2x}} + 5$ is the graph of $g(x) = \dfrac{4}{1 + 6e^{-2x}}$ shifted to the right five units.

64. The graph of a Gaussian model will never have an x-intercept.

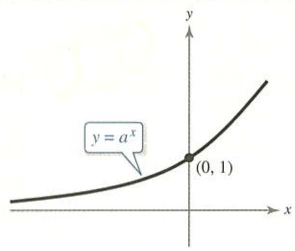
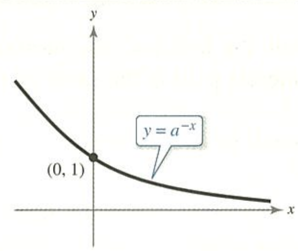
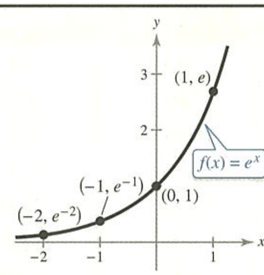
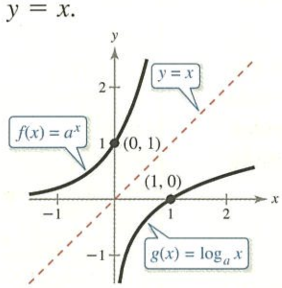
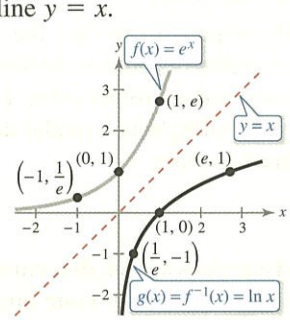
65. Writing Use your school's library, the Internet, or some other reference source to write a paper describing John Napier's work with logarithms.

66. HOW DO YOU SEE IT? Identify each model as exponential growth, exponential decay, Gaussian, linear, logarithmic, logistic growth, quadratic, or none of the above. Explain your reasoning.



Project: Sales per Share To work an extended application analyzing the sales per share for Kohl's Corporation from 1999 through 2014, visit this text's website at *LarsonPrecalculus.com.* *(Source: Kohl's Corporation)*

Chapter Summary

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 3.1	Recognize and evaluate exponential functions with base a *(p. 198)*.	The exponential function f with base a is denoted by $f(x) = a^x$, where $a > 0$, $a \neq 1$, and x is any real number.	1–6
	Graph exponential functions and use a One-to-One Property *(p. 199)*.	  **One-to-One Property:** For $a > 0$ and $a \neq 1$, $a^x = a^y$ if and only if $x = y$.	7–20
	Recognize, evaluate, and graph exponential functions with base e *(p. 202)*.	The function $f(x) = e^x$ is called the natural exponential function. 	21–28
	Use exponential functions to model and solve real-life problems *(p. 203)*.	Exponential functions are used in compound interest formulas (see Example 8) and in radioactive decay models (see Example 9).	29–32
Section 3.2	Recognize and evaluate logarithmic functions with base a *(p. 209)*.	For $x > 0$, $a > 0$, and $a \neq 1$, $y = \log_a x$ if and only if $x = a^y$. The function $f(x) = \log_a x$ is the logarithmic function with base a. The logarithmic function with base 10 is called the common logarithmic function. It is denoted by $\log_{10}$ or log.	33–44
	Graph logarithmic functions *(p. 211)*, and recognize, evaluate, and graph natural logarithmic functions *(p. 213)*.	The graph of $g(x) = \log_a x$ is a reflection of the graph of $f(x) = a^x$ in the line $y = x$.  The function $g(x) = \ln x$, $x > 0$, is called the natural logarithmic function. Its graph is a reflection of the graph of $f(x) = e^x$ in the line $y = x$. 	45–56
	Use logarithmic functions to model and solve real-life problems *(p. 215)*.	A logarithmic function can model human memory. (See Example 11.)	57, 58

	What Did You Learn?	Explanation/Examples	Review Exercises
Section 3.3	Use the change-of-base formula to rewrite and evaluate logarithmic expressions *(p. 219)*.	Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as follows. **Base *b***: $\log_a x = \dfrac{\log_b x}{\log_b a}$ **Base 10**: $\log_a x = \dfrac{\log x}{\log a}$ **Base *e***: $\log_a x = \dfrac{\ln x}{\ln a}$	59–62
	Use properties of logarithms to evaluate, rewrite, expand, or condense logarithmic expressions *(pp. 220–221)*.	Let a be a positive number such that $a \neq 1$, let n be a real number, and let u and v be positive real numbers. **1. Product Property:** $\log_a(uv) = \log_a u + \log_a v$ $\ln(uv) = \ln u + \ln v$ **2. Quotient Property:** $\log_a(u/v) = \log_a u - \log_a v$ $\ln(u/v) = \ln u - \ln v$ **3. Power Property:** $\log_a u^n = n \log_a u$, $\ln u^n = n \ln u$	63–78
	Use logarithmic functions to model and solve real-life problems *(p. 222)*.	Logarithmic functions can help you find an equation that relates the periods of several planets and their distances from the sun. (See Example 7.)	79, 80
Section 3.4	Solve simple exponential and logarithmic equations *(p. 226)*.	One-to-One Properties and Inverse Properties of exponential or logarithmic functions are used to solve exponential or logarithmic equations.	81–86
	Solve more complicated exponential equations *(p. 227)* and logarithmic equations *(p. 229)*.	To solve more complicated equations, rewrite the equations to allow the use of the One-to-One Properties or Inverse Properties of exponential or logarithmic functions. (See Examples 2–9.)	87–102
	Use exponential and logarithmic equations to model and solve real-life problems *(p. 231)*.	Exponential and logarithmic equations can help you determine how long it will take to double an investment (see Example 10) and find the year in which an industry had a given amount of sales (see Example 11).	103, 104
Section 3.5	Recognize the five most common types of models involving exponential and logarithmic functions *(p. 236)*.	**1. Exponential growth model:** $y = ae^{bx}, \quad b > 0$ **2. Exponential decay model:** $y = ae^{-bx}, \quad b > 0$ **3. Gaussian model:** $y = ae^{-(x-b)^2/c}$ **4. Logistic growth model:** $y = \dfrac{a}{1 + be^{-rx}}$ **5. Logarithmic models:** $y = a + b \ln x$, $y = a + b \log x$	105–110
	Use exponential growth and decay functions to model and solve real-life problems *(p. 237)*.	An exponential growth function can help you model a population of fruit flies (see Example 2), and an exponential decay function can help you estimate the age of a fossil (see Example 3).	111, 112
	Use Gaussian functions *(p. 240)*, logistic growth functions *(p. 241)*, and logarithmic functions *(p. 242)* to model and solve real-life problems.	A Gaussian function can help you model SAT mathematics scores for college-bound seniors. (See Example 4.) A logistic growth function can help you model the spread of a flu virus. (See Example 5.) A logarithmic function can help you find the intensity of an earthquake given its magnitude. (See Example 6.)	113–115

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

3.1 Evaluating an Exponential Function In Exercises 1–6, evaluate the function at the given value of x. Round your result to three decimal places.

1. $f(x) = 0.3^x, \quad x = 1.5$
2. $f(x) = 30^x, \quad x = \sqrt{3}$
3. $f(x) = 2^x, \quad x = \frac{2}{3}$
4. $f(x) = \left(\frac{1}{2}\right)^{2x}, \quad x = \pi$
5. $f(x) = 7(0.2^x), \quad x = -\sqrt{11}$
6. $f(x) = -14(5^x), \quad x = -0.8$

Graphing an Exponential Function In Exercises 7–12, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function.

7. $f(x) = 4^{-x} + 4$
8. $f(x) = 2.65^{x-1}$
9. $f(x) = 5^{x-2} + 4$
10. $f(x) = 2^{x-6} - 5$
11. $f(x) = \left(\frac{1}{2}\right)^{-x} + 3$
12. $f(x) = \left(\frac{1}{8}\right)^{x+2} - 5$

Using a One-to-One Property In Exercises 13–16, use a One-to-One Property to solve the equation for x.

13. $\left(\frac{1}{3}\right)^{x-3} = 9$
14. $3^{x+3} = \frac{1}{81}$
15. $e^{3x-5} = e^7$
16. $e^{8-2x} = e^{-3}$

Transforming the Graph of an Exponential Function In Exercises 17–20, describe the transformation of the graph of f that yields the graph of g.

17. $f(x) = 5^x, \quad g(x) = 5^x + 1$
18. $f(x) = 6^x, \quad g(x) = 6^{x+1}$
19. $f(x) = 3^x, \quad g(x) = 1 - 3^x$
20. $f(x) = \left(\frac{1}{2}\right)^x, \quad g(x) = -\left(\frac{1}{2}\right)^{x+2}$

Evaluating the Natural Exponential Function In Exercises 21–24, evaluate $f(x) = e^x$ at the given value of x. Round your result to three decimal places.

21. $x = 3.4$
22. $x = -2.5$
23. $x = \frac{3}{5}$
24. $x = \frac{2}{7}$

Graphing a Natural Exponential Function In Exercises 25–28, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function.

25. $h(x) = e^{-x/2}$
26. $h(x) = 2 - e^{-x/2}$
27. $f(x) = e^{x+2}$
28. $s(t) = 4e^{t-1}$

29. **Waiting Times** The average time between new posts on a message board is 3 minutes. The probability F of waiting less than t minutes until the next post is approximated by the model $F(t) = 1 - e^{-t/3}$. A message has just been posted. Find the probability that the next post will be within (a) 1 minute, (b) 2 minutes, and (c) 5 minutes.

30. **Depreciation** After t years, the value V of a car that originally cost \$23,970 is given by $V(t) = 23{,}970\left(\frac{3}{4}\right)^t$.
    (a) Use a graphing utility to graph the function.
    (b) Find the value of the car 2 years after it was purchased.
    (c) According to the model, when does the car depreciate most rapidly? Is this realistic? Explain.
    (d) According to the model, when will the car have no value?

Compound Interest In Exercises 31 and 32, complete the table by finding the balance A when P dollars is invested at rate r for t years and compounded n times per year.

n	1	2	4	12	365	Continuous
A						

31. $P = \$5000, \quad r = 3\%, \quad t = 10$ years
32. $P = \$4500, \quad r = 2.5\%, \quad t = 30$ years

3.2 Writing a Logarithmic Equation In Exercises 33–36, write the exponential equation in logarithmic form. For example, the logarithmic form of $2^3 = 8$ is $\log_2 8 = 3$.

33. $3^3 = 27$
34. $25^{3/2} = 125$
35. $e^{0.8} = 2.2255 \ldots$
36. $e^0 = 1$

Evaluating a Logarithm In Exercises 37–40, evaluate the logarithm at the given value of x without using a calculator.

37. $f(x) = \log x, \quad x = 1000$
38. $g(x) = \log_9 x, \quad x = 3$
39. $g(x) = \log_2 x, \quad x = \frac{1}{4}$
40. $f(x) = \log_3 x, \quad x = \frac{1}{81}$

Using a One-to-One Property In Exercises 41–44, use a One-to-One Property to solve the equation for x.

41. $\log_4(x + 7) = \log_4 14$
42. $\log_8(3x - 10) = \log_8 5$
43. $\ln(x + 9) = \ln 4$
44. $\log(3x - 2) = \log 7$

Sketching the Graph of a Logarithmic Function In Exercises 45–48, find the domain, x-intercept, and vertical asymptote of the logarithmic function and sketch its graph.

45. $g(x) = \log_7 x$
46. $f(x) = \log \dfrac{x}{3}$
47. $f(x) = 4 - \log(x + 5)$
48. $f(x) = \log(x - 3) + 1$

Evaluating a Logarithmic Function **In Exercises 49–52, use a calculator to evaluate the function at the given value of x. Round your result to three decimal places, if necessary.**

49. $f(x) = \ln x,\ x = 22.6$

50. $f(x) = \ln x,\ x = e^{-12}$

51. $f(x) = \frac{1}{2}\ln x,\ x = \sqrt{e}$

52. $f(x) = 5\ln x,\ x = 0.98$

Graphing a Natural Logarithmic Function **In Exercises 53–56, find the domain, x-intercept, and vertical asymptote of the logarithmic function and sketch its graph.**

53. $f(x) = \ln x + 6$

54. $f(x) = \ln x - 5$

55. $h(x) = \ln(x - 6)$

56. $f(x) = \ln(x + 4)$

57. Astronomy The formula $M = m - 5\log(d/10)$ gives the distance d (in parsecs) from Earth to a star with apparent magnitude m and absolute magnitude M. The star Rasalhague has an apparent magnitude of 2.08 and an absolute magnitude of 1.3. Find the distance from Earth to Rasalhague.

58. Snow Removal The number of miles s of roads cleared of snow is approximated by the model

$$s = 25 - \frac{13\ln(h/12)}{\ln 3},\quad 2 \le h \le 15$$

where h is the depth (in inches) of the snow. Use this model to find s when $h = 10$ inches.

3.3 Using the Change-of-Base Formula **In Exercises 59–62, evaluate the logarithm using the change-of-base formula (a) with common logarithms and (b) with natural logarithms. Round your results to three decimal places.**

59. $\log_2 6$

60. $\log_{12} 200$

61. $\log_{1/2} 5$

62. $\log_4 0.75$

Using Properties of Logarithms **In Exercises 63–66, use the properties of logarithms to write the logarithm in terms of $\log_2 3$ and $\log_2 5$.**

63. $\log_2 \frac{5}{3}$

64. $\log_2 45$

65. $\log_2 \frac{9}{5}$

66. $\log_2 \frac{20}{9}$

Expanding a Logarithmic Expression **In Exercises 67–72, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)**

67. $\log 7x^2$

68. $\log 11x^3$

69. $\log_3 \dfrac{9}{\sqrt{x}}$

70. $\log_7 \dfrac{\sqrt[3]{x}}{19}$

71. $\ln x^2y^2z$

72. $\ln\left(\dfrac{y-1}{3}\right)^2,\quad y > 1$

Condensing a Logarithmic Expression **In Exercises 73–78, condense the expression to the logarithm of a single quantity.**

73. $\ln 7 + \ln x$

74. $\log_2 y - \log_2 3$

75. $\log x - \frac{1}{2}\log y$

76. $3\ln x + 2\ln(x + 1)$

77. $\frac{1}{2}\log_3 x - 2\log_3(y + 8)$

78. $5\ln(x - 2) - \ln(x + 2) - 3\ln x$

79. Climb Rate The time t (in minutes) for a small plane to climb to an altitude of h feet is modeled by

$$t = 50\log[18{,}000/(18{,}000 - h)]$$

where 18,000 feet is the plane's absolute ceiling.

(a) Determine the domain of the function in the context of the problem.

(b) Use a graphing utility to graph the function and identify any asymptotes.

(c) As the plane approaches its absolute ceiling, what can be said about the time required to increase its altitude?

(d) Find the time it takes for the plane to climb to an altitude of 4000 feet.

80. Human Memory Model Students in a learning theory study took an exam and then retested monthly for 6 months with an equivalent exam. The data obtained in the study are given by the ordered pairs (t, s), where t is the time (in months) after the initial exam and s is the average score for the class. Use the data to find a logarithmic equation that relates t and s.

(1, 84.2), (2, 78.4), (3, 72.1),
(4, 68.5), (5, 67.1), (6, 65.3)

3.4 Solving a Simple Equation **In Exercises 81–86, solve for x.**

81. $5^x = 125$

82. $6^x = \frac{1}{216}$

83. $e^x = 3$

84. $\log x - \log 5 = 0$

85. $\ln x = 4$

86. $\ln x = -1.6$

Solving an Exponential Equation **In Exercises 87–90, solve the exponential equation algebraically. Approximate the result to three decimal places.**

87. $e^{4x} = e^{x^2+3}$

88. $e^{3x} = 25$

89. $2^x - 3 = 29$

90. $e^{2x} - 6e^x + 8 = 0$

Solving a Logarithmic Equation In Exercises 91–98, solve the logarithmic equation algebraically. Approximate the result to three decimal places.

91. $\ln 3x = 8.2$

92. $4 \ln 3x = 15$

93. $\ln x + \ln(x - 3) = 1$

94. $\ln(x + 2) - \ln x = 2$

95. $\log_8(x - 1) = \log_8(x - 2) - \log_8(x + 2)$

96. $\log_6(x + 2) - \log_6 x = \log_6(x + 5)$

97. $\log(1 - x) = -1$

98. $\log(-x - 4) = 2$

Using Technology In Exercises 99–102, use a graphing utility to graphically solve the equation. Approximate the result to three decimal places. Verify your result algebraically.

99. $25e^{-0.3x} = 12$

100. $2 = 5 - e^{x+7}$

101. $2 \ln(x + 3) - 3 = 0$

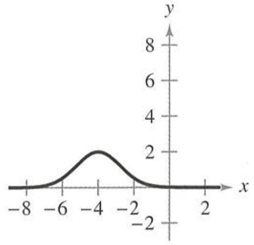
102. $2 \ln x - \ln(3x - 1) = 0$

103. Compound Interest You deposit \$8500 in an account that pays 1.5% interest, compounded continuously. How long will it take for the money to triple?

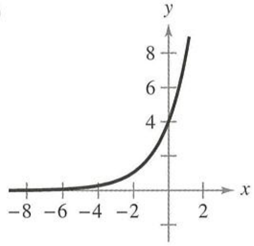
104. Meteorology The speed of the wind S (in miles per hour) near the center of a tornado and the distance d (in miles) the tornado travels are related by the model $S = 93 \log d + 65$. On March 18, 1925, a large tornado struck portions of Missouri, Illinois, and Indiana with a wind speed at the center of about 283 miles per hour. Approximate the distance traveled by this tornado.

3.5 Matching a Function with Its Graph In Exercises 105–110, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

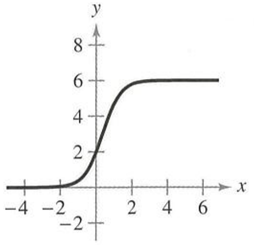
(a)



(b)



(c)



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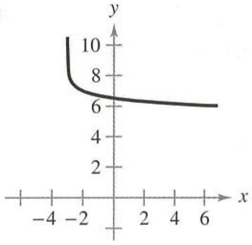

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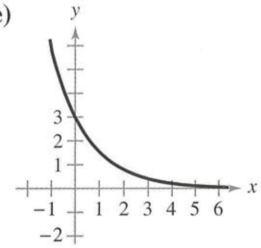
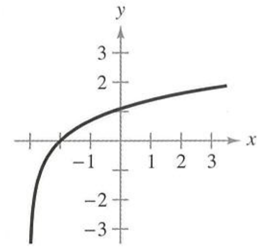


(f)



105. $y = 3e^{-2x/3}$

106. $y = 4e^{2x/3}$

107. $y = \ln(x + 3)$

108. $y = 7 - \log(x + 3)$

109. $y = 2e^{-(x+4)^2/3}$

110. $y = \dfrac{6}{1 + 2e^{-2x}}$

111. Finding an Exponential Model Find the exponential model $y = ae^{bx}$ that fits the points (0, 2) and (4, 3).

112. Wildlife Population A species of bat is in danger of becoming extinct. Five years ago, the total population of the species was 2000. Two years ago, the total population of the species was 1400. What was the total population of the species one year ago?

113. Test Scores The test scores for a biology test follow the normal distribution

$$y = 0.0499e^{-(x-71)^2/128}, \quad 40 \le x \le 100$$

where x is the test score. Use a graphing utility to graph the equation and estimate the average test score.

114. Typing Speed In a typing class, the average number N of words per minute typed after t weeks of lessons is

$$N = 157/(1 + 5.4e^{-0.12t}).$$

Find the time necessary to type (a) 50 words per minute and (b) 75 words per minute.

115. Sound Intensity The relationship between the number of decibels β and the intensity of a sound I (in watts per square meter) is

$$\beta = 10 \log(I/10^{-12}).$$

Find the intensity I for each decibel level β.

(a) $\beta = 60$ (b) $\beta = 135$ (c) $\beta = 1$

Exploration

116. Graph of an Exponential Function Consider the graph of $y = e^{kt}$. Describe the characteristics of the graph when k is positive and when k is negative.

True or False? In Exercises 117 and 118, determine whether the equation is true or false. Justify your answer.

117. $\log_b b^{2x} = 2x$

118. $\ln(x + y) = \ln x + \ln y$

Chapter Test

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

In Exercises 1–4, evaluate the expression. Round your result to three decimal places.

1. $0.7^{2.5}$ **2.** $3^{-\pi}$ **3.** $e^{-7/10}$ **4.** $e^{3.1}$

In Exercises 5–7, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function.

5. $f(x) = 10^{-x}$ **6.** $f(x) = -6^{x-2}$ **7.** $f(x) = 1 - e^{2x}$

8. Evaluate (a) $\log_7 7^{-0.89}$ and (b) $4.6 \ln e^2$.

In Exercises 9–11, find the domain, *x*-intercept, and vertical asymptote of the logarithmic function and sketch its graph.

9. $f(x) = 4 + \log x$ **10.** $f(x) = \ln(x - 4)$ **11.** $f(x) = 1 + \ln(x + 6)$

In Exercises 12–14, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

12. $\log_5 35$ **13.** $\log_{16} 0.63$ **14.** $\log_{3/4} 24$

In Exercises 15–17, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)

15. $\log_2 3a^4$ **16.** $\ln \dfrac{\sqrt{x}}{7}$ **17.** $\log \dfrac{10x^2}{y^3}$

In Exercises 18–20, condense the expression to the logarithm of a single quantity.

18. $\log_3 13 + \log_3 y$ **19.** $4 \ln x - 4 \ln y$

20. $3 \ln x - \ln(x + 3) + 2 \ln y$

In Exercises 21–26, solve the equation algebraically. Approximate the result to three decimal places, if necessary.

21. $5^x = \dfrac{1}{25}$ **22.** $3e^{-5x} = 132$

23. $\dfrac{1025}{8 + e^{4x}} = 5$ **24.** $\ln x = \dfrac{1}{2}$

25. $18 + 4 \ln x = 7$ **26.** $\log x + \log(x - 15) = 2$

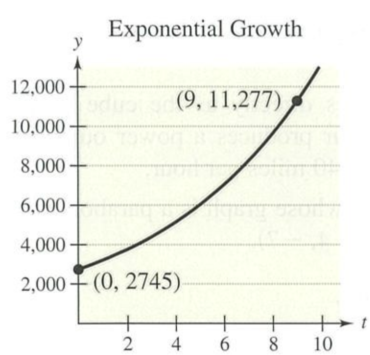


Figure for 27

27. Find the exponential growth model that fits the points shown in the graph.

28. The half-life of radioactive actinium (^{227}Ac) is 21.77 years. What percent of a present amount of radioactive actinium will remain after 19 years?

29. A model that can predict a child's height H (in centimeters) based on the child's age is $H = 70.228 + 5.104x + 9.222 \ln x$, $\frac{1}{4} \le x \le 6$, where x is the child's age in years. *(Source: Snapshots of Applications in Mathematics)*

(a) Construct a table of values for the model. Then sketch the graph of the model.

(b) Use the graph from part (a) to predict the height of a four-year-old child. Then confirm your prediction algebraically.

Cumulative Test for Chapters 1–3

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Take this test as you would take a test in class. When you are finished, check your work against the answers given in the back of the book.

1. Plot the points $(-2, 5)$ and $(3, -1)$. Find the midpoint of the line segment joining the points and the distance between the points.

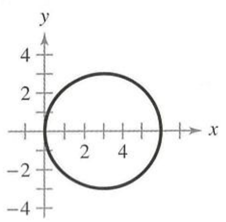


Figure for 6

In Exercises 2–4, sketch the graph of the equation.

2. $x - 3y + 12 = 0$ **3.** $y = x^2 - 9$ **4.** $y = \sqrt{4 - x}$

5. Find the slope-intercept form of the equation of the line passing through $\left(-\frac{1}{2}, 1\right)$ and $(3, 8)$.

6. Explain why the graph at the left does not represent y as a function of x.

7. Let $f(x) = \dfrac{x}{x - 2}$. Find each function value, if possible.

(a) $f(6)$ (b) $f(2)$ (c) $f(s + 2)$

8. Compare the graph of each function with the graph of $y = \sqrt[3]{x}$. (*Note:* It is not necessary to sketch the graphs.)

(a) $r(x) = \frac{1}{2}\sqrt[3]{x}$ (b) $h(x) = \sqrt[3]{x} + 2$ (c) $g(x) = \sqrt[3]{x + 2}$

In Exercises 9 and 10, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?

9. $f(x) = x - 4, \quad g(x) = 3x + 1$

10. $f(x) = \sqrt{x - 1}, \quad g(x) = x^2 + 1$

In Exercises 11 and 12, find (a) $f \circ g$ and (b) $g \circ f$. Find the domain of each composite function.

11. $f(x) = 2x^2, \quad g(x) = \sqrt{x + 6}$

12. $f(x) = x - 2, \quad g(x) = |x|$

13. Determine whether $h(x) = 3x - 4$ has an inverse function. If is does, find the inverse function.

14. The power P produced by a wind turbine varies directly as the cube of the wind speed S. A wind speed of 27 miles per hour produces a power output of 750 kilowatts. Find the output for a wind speed of 40 miles per hour.

15. Write the standard form of the quadratic function whose graph is a parabola with vertex $(-8, 5)$ and that passes through the point $(-4, -7)$.

In Exercises 16–18, sketch the graph of the function.

16. $h(x) = -x^2 + 10x - 21$

17. $f(t) = -\dfrac{1}{2}(t - 1)^2(t + 2)^2$

18. $g(s) = s^3 - 3s^2$

In Exercises 19–21, find all the zeros of the function.

19. $f(x) = x^3 + 2x^2 + 4x + 8$

20. $f(x) = x^4 + 4x^3 - 21x^2$

21. $f(x) = 2x^4 - 11x^3 + 30x^2 - 62x - 40$

22. Use long division to divide: $\dfrac{6x^3 - 4x^2}{2x^2 + 1}$.

23. Use synthetic division to divide $3x^4 + 2x^2 - 5x + 3$ by $x - 2$.

24. Use the Intermediate Value Theorem and the *table* feature of a graphing utility to find an interval one unit in length in which the function $g(x) = x^3 + 3x^2 - 6$ is guaranteed to have a zero. Then adjust the table to approximate the real zero to the nearest thousandth.

In Exercises 25–27, sketch the graph of the rational function. Identify all intercepts and find any asymptotes.

25. $f(x) = \dfrac{2x}{x^2 + 2x - 3}$

26. $f(x) = \dfrac{x^2 - 4}{x^2 + x - 2}$

27. $f(x) = \dfrac{x^3 - 2x^2 - 9x + 18}{x^2 + 4x + 3}$

In Exercises 28 and 29, solve the inequality. Then graph the solution set.

28. $2x^3 - 18x \leq 0$

29. $\dfrac{1}{x + 1} \geq \dfrac{1}{x + 5}$

In Exercises 30 and 31, describe the transformations of the graph of f that yield the graph of g.

30. $f(x) = \left(\frac{2}{5}\right)^x, \quad g(x) = -\left(\frac{2}{5}\right)^{-x+3}$

31. $f(x) = 2.2^x, \quad g(x) = -2.2^x + 4$

In Exercises 32–35, use a calculator to evaluate the expression. Round your result to three decimal places.

32. $\log 98$

33. $\log \frac{6}{7}$

34. $\ln \sqrt{31}$

35. $\ln\left(\sqrt{30} - 4\right)$

36. Use the properties of logarithms to expand $\ln\left(\dfrac{x^2 - 25}{x^4}\right)$, where $x > 5$.

37. Condense $2 \ln x - \frac{1}{2} \ln(x + 5)$ to the logarithm of a single quantity.

In Exercises 38–40, solve the equation algebraically. Approximate the result to three decimal places.

38. $6e^{2x} = 72$

39. $e^{2x} - 13e^x + 42 = 0$

40. $\ln \sqrt{x + 2} = 3$

41. On the day a grandchild is born, a grandparent deposits \$2500 in a fund earning 7.5% interest, compounded continuously. Determine the balance in the account on the grandchild's 25th birthday.

42. The number N of bacteria in a culture is given by the model $N = 175e^{kt}$, where t is the time in hours. If $N = 420$ when $t = 8$, then estimate the time required for the population to double in size.

43. The population P (in millions) of Texas from 2001 through 2014 can be approximated by the model $P = 20.913e^{0.0184t}$, where t represents the year, with $t = 1$ corresponding to 2001. According to this model, when will the population reach 32 million? *(Source: U.S. Census Bureau)*

Proofs in Mathematics

Each of the three properties of logarithms listed below can be proved by using properties of exponential functions.

SLIDE RULES

William Oughtred (1574–1660) is credited with inventing the slide rule. The slide rule is a computational device with a sliding portion and a fixed portion. A slide rule enables you to perform multiplication by using the Product Property of Logarithms. There are other slide rules that allow for the calculation of roots and trigonometric functions. Mathematicians and engineers used slide rules until the hand-held calculator came into widespread use in the 1970s.

Properties of Logarithms *(p. 220)*

Let a be a positive number such that $a \neq 1$, let n be a real number, and let u and v be positive real numbers.

	Logarithm with Base a	Natural Logarithm
1. **Product Property:**	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
2. **Quotient Property:**	$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$
3. **Power Property:**	$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$

Proof

Let

$$x = \log_a u \quad \text{and} \quad y = \log_a v.$$

The corresponding exponential forms of these two equations are

$$a^x = u \quad \text{and} \quad a^y = v.$$

To prove the Product Property, multiply u and v to obtain

$$\begin{aligned} uv &= a^x a^y \\ &= a^{x+y}. \end{aligned}$$

The corresponding logarithmic form of $uv = a^{x+y}$ is $\log_a(uv) = x + y$. So,

$$\log_a(uv) = \log_a u + \log_a v.$$

To prove the Quotient Property, divide u by v to obtain

$$\begin{aligned} \frac{u}{v} &= \frac{a^x}{a^y} \\ &= a^{x-y}. \end{aligned}$$

The corresponding logarithmic form of $\frac{u}{v} = a^{x-y}$ is $\log_a \frac{u}{v} = x - y$. So,

$$\log_a \frac{u}{v} = \log_a u - \log_a v.$$

To prove the Power Property, substitute a^x for u in the expression $\log_a u^n$.

$$\begin{aligned} \log_a u^n &= \log_a (a^x)^n && \text{Substitute } a^x \text{ for } u. \\ &= \log_a a^{nx} && \text{Property of Exponents} \\ &= nx && \text{Inverse Property} \\ &= n \log_a u && \text{Substitute } \log_a u \text{ for } x. \end{aligned}$$

So, $\log_a u^n = n \log_a u$.

P.S. Problem Solving

1. **Graphical Reasoning** Graph the exponential function $y = a^x$ for $a = 0.5, 1.2$, and 2.0. Which of these curves intersects the line $y = x$? Determine all positive numbers a for which the curve $y = a^x$ intersects the line $y = x$.

2. **Graphical Reasoning** Use a graphing utility to graph each of the functions $y_1 = e^x$, $y_2 = x^2$, $y_3 = x^3$, $y_4 = \sqrt{x}$, and $y_5 = |x|$. Which function increases at the greatest rate as x approaches ∞?

3. **Conjecture** Use the result of Exercise 2 to make a conjecture about the rate of growth of $y_1 = e^x$ and $y = x^n$, where n is a natural number and x approaches ∞.

4. **Implication of "Growing Exponentially"** Use the results of Exercises 2 and 3 to describe what is implied when it is stated that a quantity is growing exponentially.

5. **Exponential Function** Given the exponential function

$$f(x) = a^x$$

show that

(a) $f(u + v) = f(u) \cdot f(v)$ and (b) $f(2x) = [f(x)]^2$.

6. **Hyperbolic Functions** Given that

$$f(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad g(x) = \frac{e^x - e^{-x}}{2}$$

show that

$$[f(x)]^2 - [g(x)]^2 = 1.$$

7. **Graphical Reasoning** Use a graphing utility to compare the graph of the function $y = e^x$ with the graph of each function. [$n!$ (read "n factorial") is defined as $n! = 1 \cdot 2 \cdot 3 \cdot \cdot \cdot (n-1) \cdot n$.]

(a) $y_1 = 1 + \dfrac{x}{1!}$

(b) $y_2 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!}$

(c) $y_3 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}$

8. **Identifying a Pattern** Identify the pattern of successive polynomials given in Exercise 7. Extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = e^x$. What do you think this pattern implies?

9. **Finding an Inverse Function** Graph the function

$$f(x) = e^x - e^{-x}.$$

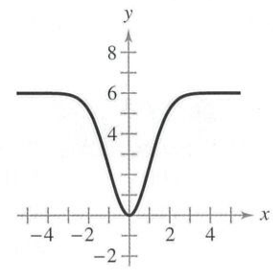
From the graph, the function appears to be one-to-one. Assume that f has an inverse function and find $f^{-1}(x)$.

10. **Finding a Pattern for an Inverse Function** Find a pattern for $f^{-1}(x)$ when

$$f(x) = \frac{a^x + 1}{a^x - 1}$$

where $a > 0$, $a \neq 1$.

11. **Determining the Equation of a Graph** Determine whether the graph represents equation (a), (b), or (c). Explain your reasoning.



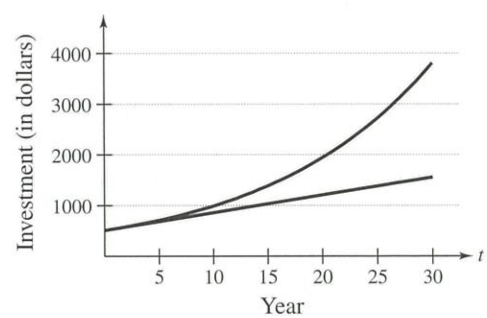
(a) $y = 6e^{-x^2/2}$

(b) $y = \dfrac{6}{1 + e^{-x/2}}$

(c) $y = 6(1 - e^{-x^2/2})$

12. **Simple and Compound Interest** You have two options for investing \$500. The first earns 7% interest compounded annually, and the second earns 7% simple interest. The figure shows the growth of each investment over a 30-year period.

(a) Determine which graph represents each type of investment. Explain your reasoning.



(b) Verify your answer in part (a) by finding the equations that model the investment growth and by graphing the models.

(c) Which option would you choose? Explain.

13. **Radioactive Decay** Two different samples of radioactive isotopes are decaying. The isotopes have initial amounts of c_1 and c_2 and half-lives of k_1 and k_2, respectively. Find an expression for the time t required for the samples to decay to equal amounts.

14. **Bacteria Decay** A lab culture initially contains 500 bacteria. Two hours later, the number of bacteria decreases to 200. Find the exponential decay model of the form

$$B = B_0 a^{kt}$$

that approximates the number of bacteria B in the culture after t hours.

15. **Colonial Population** The table shows the colonial population estimates of the American colonies for each decade from 1700 through 1780. *(Source: U.S. Census Bureau)*

DATA

Year	Population
1700	250,900
1710	331,700
1720	466,200
1730	629,400
1740	905,600
1750	1,170,800
1760	1,593,600
1770	2,148,100
1780	2,780,400

Spreadsheet at LarsonPrecalculus.com

Let y represent the population in the year t, with $t = 0$ corresponding to 1700.

(a) Use the *regression* feature of a graphing utility to find an exponential model for the data.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to plot the data and the models from parts (a) and (b) in the same viewing window.

(d) Which model is a better fit for the data? Would you use this model to predict the population of the United States in 2020? Explain your reasoning.

16. **Ratio of Logarithms** Show that

$$\frac{\log_a x}{\log_{a/b} x} = 1 + \log_a \frac{1}{b}.$$

17. **Solving a Logarithmic Equation** Solve

$$(\ln x)^2 = \ln x^2.$$

18. **Graphical Reasoning** Use a graphing utility to compare the graph of each function with the graph of $y = \ln x$.

(a) $y_1 = x - 1$

(b) $y_2 = (x - 1) - \frac{1}{2}(x - 1)^2$

(c) $y_3 = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3$

19. **Identifying a Pattern** Identify the pattern of successive polynomials given in Exercise 18. Extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = \ln x$. What do you think the pattern implies?

20. **Finding Slope and y-Intercept** Take the natural log of each side of each equation below.

$$y = ab^x, \quad y = ax^b$$

(a) What are the slope and y-intercept of the line relating x and $\ln y$ for $y = ab^x$?

(b) What are the slope and y-intercept of the line relating $\ln x$ and $\ln y$ for $y = ax^b$?

Ventilation Rate **In Exercises 21 and 22, use the model**

$$y = 80.4 - 11 \ln x, \quad 100 \le x \le 1500$$

which approximates the minimum required ventilation rate in terms of the air space per child in a public school classroom. In the model, x is the air space (in cubic feet) per child and y is the ventilation rate (in cubic feet per minute) per child.

21. Use a graphing utility to graph the model and approximate the required ventilation rate when there are 300 cubic feet of air space per child.

22. In a classroom designed for 30 students, the air conditioning system can move 450 cubic feet of air per minute.

(a) Determine the ventilation rate per child in a full classroom.

(b) Estimate the air space required per child.

(c) Determine the minimum number of square feet of floor space required for the room when the ceiling height is 30 feet.

Using Technology **In Exercises 23–26, (a) use a graphing utility to create a scatter plot of the data, (b) decide whether the data could best be modeled by a linear model, an exponential model, or a logarithmic model, (c) explain why you chose the model you did in part (b), (d) use the *regression* feature of the graphing utility to find the model you chose in part (b) for the data and graph the model with the scatter plot, and (e) determine how well the model you chose fits the data.**

23. (1, 2.0), (1.5, 3.5), (2, 4.0), (4, 5.8), (6, 7.0), (8, 7.8)

24. (1, 4.4), (1.5, 4.7), (2, 5.5), (4, 9.9), (6, 18.1), (8, 33.0)

25. (1, 7.5), (1.5, 7.0), (2, 6.8), (4, 5.0), (6, 3.5), (8, 2.0)

26. (1, 5.0), (1.5, 6.0), (2, 6.4), (4, 7.8), (6, 8.6), (8, 9.0)